Induced actions by countable amenable groups

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That transformation is *the induced transformation on A*, denoted T_A.

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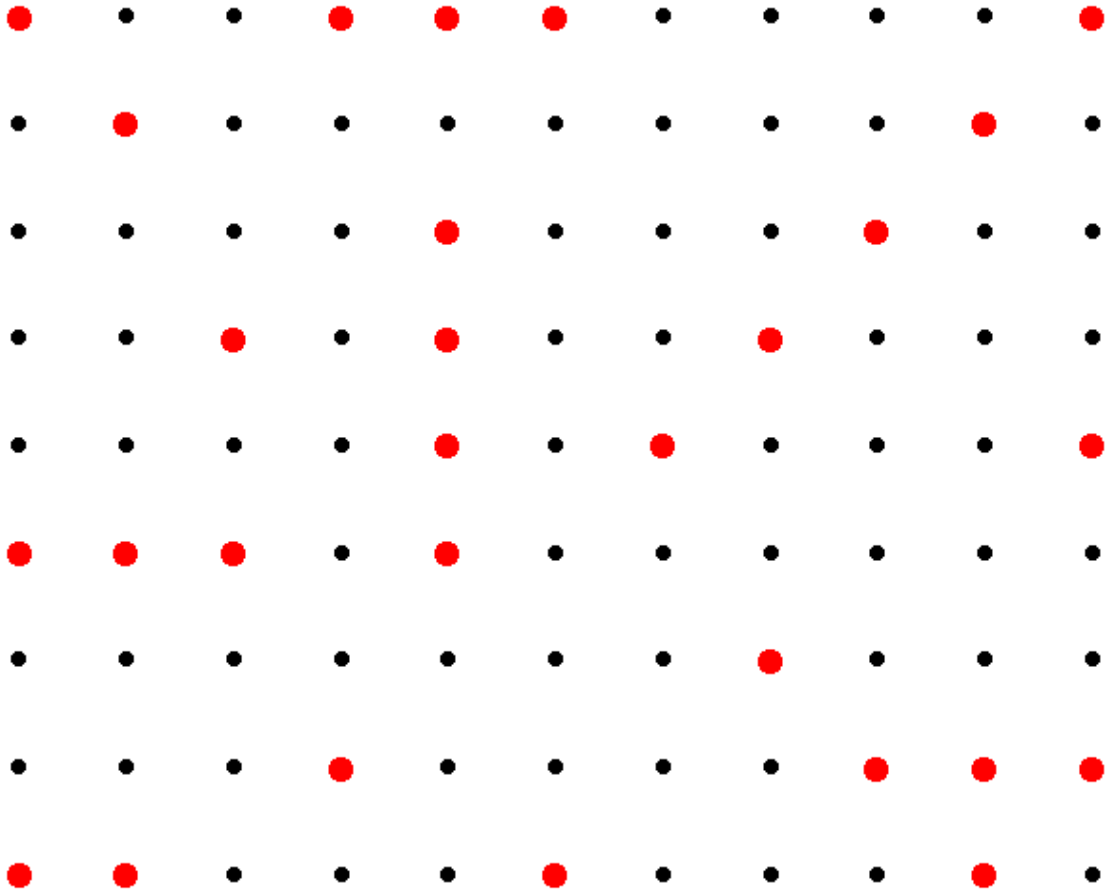
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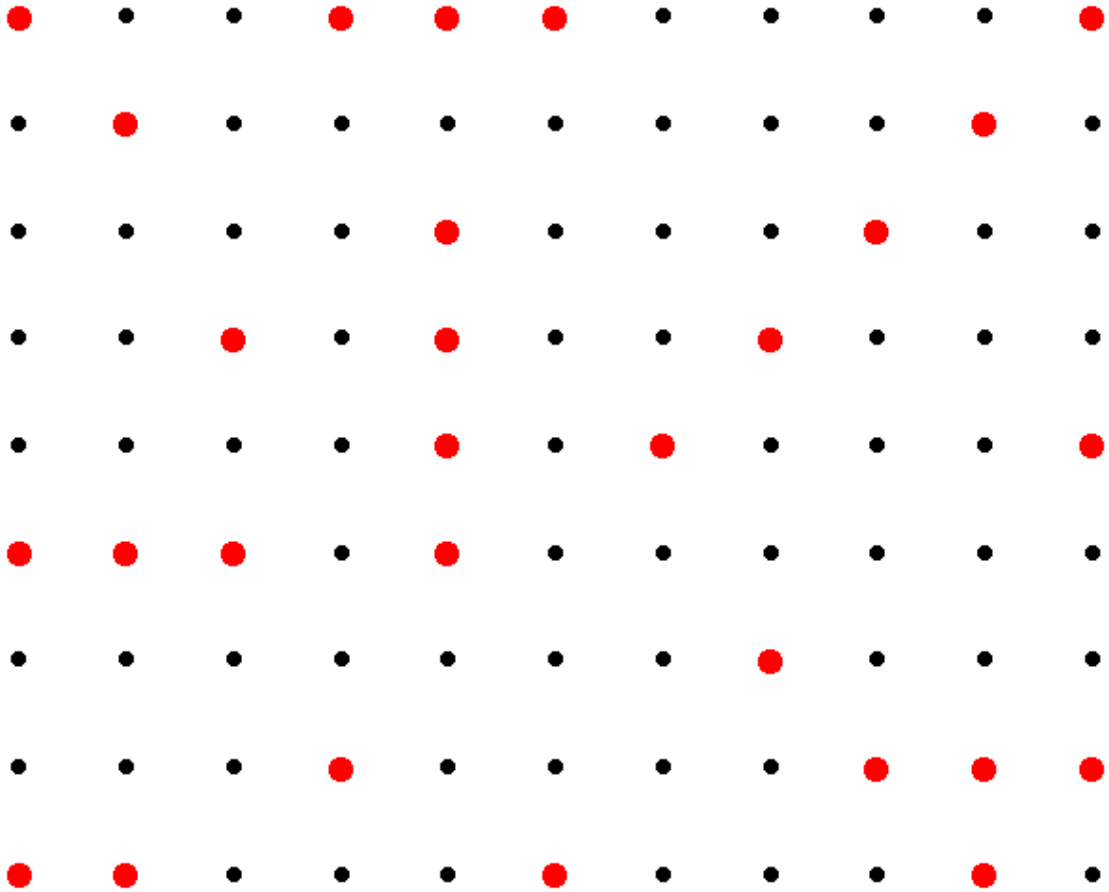
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If we assume ergodicity then we know even more: for any Følner sequence $\mathcal{F}$, almost every orbit starting from A visits A infinitely many times with positive $\mathcal{F}$-density.

But still, the times of the visits may form a subset of $\mathbb{Z}^2$ completely "incompatible" with $\mathbb{Z}^2$.





So, there seems to be no general method of turning the visits into an orbit of another $\mathbb{Z}^2$-action defined on A.

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However, every countable amenable group admits an *invariant random order*, that is, a family of linear orders equipped with a probability measure ν that is invariant under shifting the group.

Moreover, there exists such a measure supported by orders $\prec$ *of type* $\mathbb{Z}$ (i.e., such that $(G, \prec)$ is order-isomorphic to $(\mathbb{Z}, <)$).

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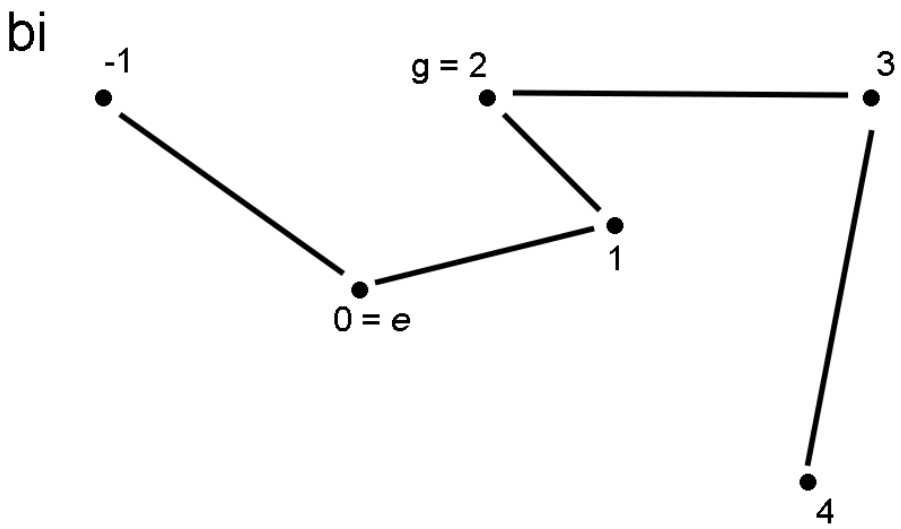
where $k \in \mathbb{Z}$ is such that $g = \mathbf{bi}(k)$. The image is again an anchored bijection.

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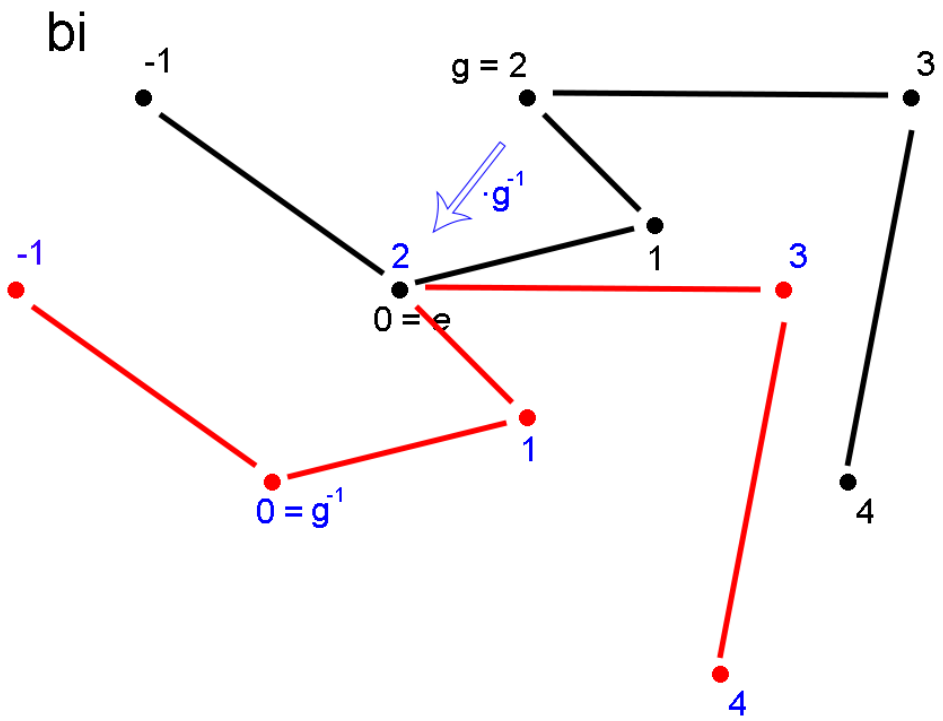
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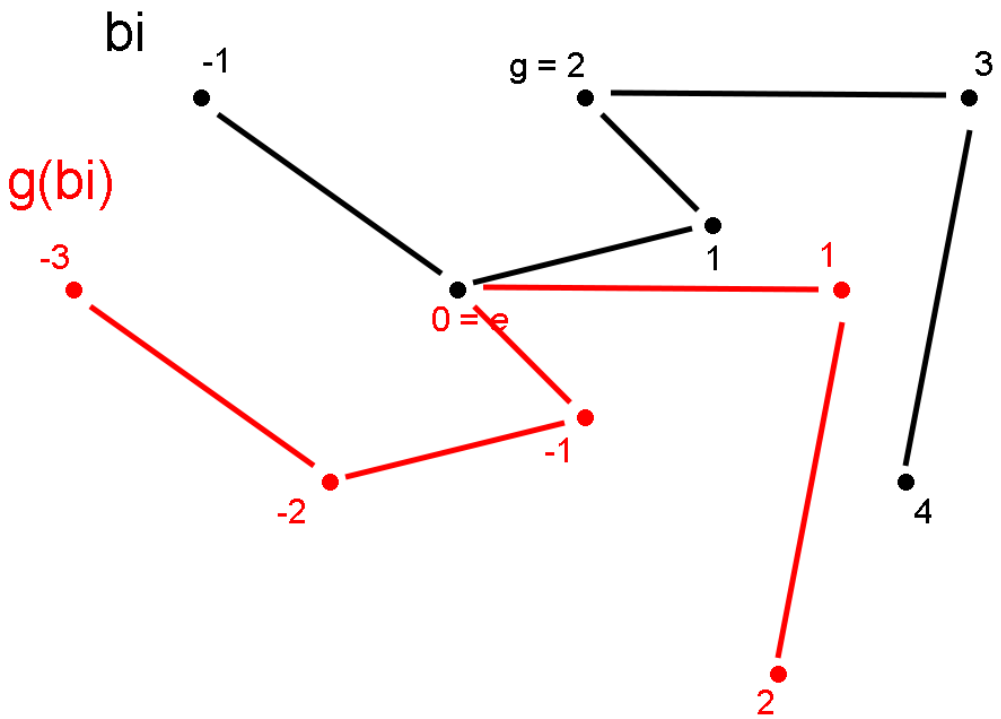
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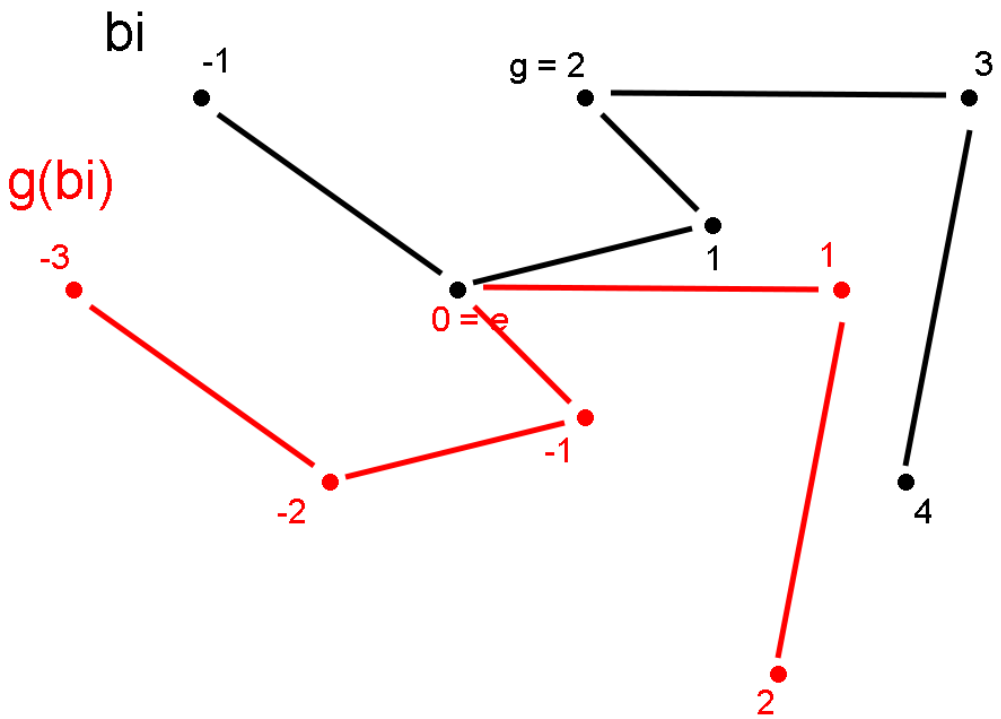
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We will denote this action of G on $\tilde{\mathcal{O}}$ by ρ.

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The quadruple (X, μ, τ, ϕ) will be called a *multiordered G-system*.

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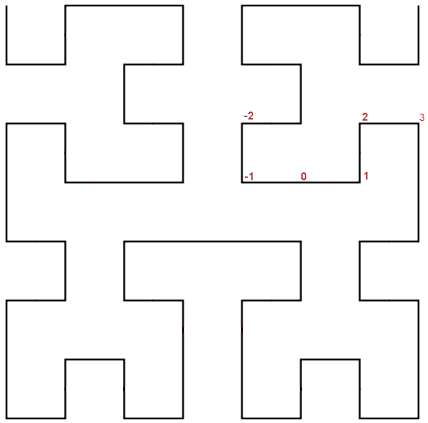
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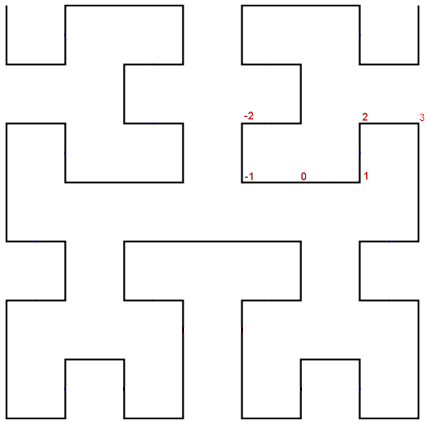


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This multiorder has the additional property of *bounded jumps*, i.e., $\mathbf{bi}(n+1) - \mathbf{bi}(n) \in F$ (F - finite).

Multiorder vs. orbit-equivalence

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(In fact, any $\mathbb{Z}$-action with the same orbits is generated by the successor map of some multiorder factor.)

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Theorem 3 ([DOWZ])

On every countable amenable group there exists a multiorder $(\tilde{\mathcal{O}}, \nu, \rho)$ with *double entropy zero*, i.e., with $h_\mu(\rho) = h_\mu(\tilde{S}) = 0$.

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Unfortunately, this method does not work. The composition rule $h_A(g_A(x)) = (hg)_A(x)$ holds only when $\phi(A)$ is invariant under ρ (for example, when the multiorder is trivial).

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Recall that the transformation on $\tilde{\mathcal{O}}$ given by $\mathbf{bi} \mapsto \mathbf{bi}(1)(\mathbf{bi})$ is the inner successor map $\tilde{S}$, so we can write

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Theorem 4

The successor map of the skew product G-action created from a $\mathbb{Z}$-system (X, μ, T) splits as the direct product of the original system (X, μ, T) with the inner successor map on the multiorder $(\tilde{\mathcal{O}}, \nu, \tilde{S})$.

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However, this time, k does not depend on x (via $g = \mathbf{bi}_x(k)$); it is selected so that $g = \mathbf{bi}(k)$, where $\mathbf{bi}$ is independent of x.

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   A similar tower construction works for multiordered G-actions. The original system can be recovered from the induced system and the return-time function evaluated for the successor map.

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