Structure theory of (multivariate) mean equicontinuous minimal systems

Tobias Jäger

Budapest-Vienna-Seminar, Krakow, 2–4 October 2026



Friedrich-Schiller-Universität Jena

Contributions

- **More than one decade of community effort . . .**
- Own contributions in joint work with
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  - María Isabel Cortez
  - Gabriel Fuhrmann
  - Felipe García-Ramos
  - Eli Glasner
  - Philipp Gohlke
  - Jaime Gomez
  - Maik Gröger
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Basic notions – univariate case

- We consider *topological dynamical systems (tds)* consisting of a compact metric space (X, d) and a homeomorphism $\varphi : X \to X$.
- All results generalise to locally compact, second countable abelian group actions, many to ameanable group actions.
- A tds (X, φ) is *equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that $d(x, y) < \delta$ implies $d\left(\varphi^n(x), \varphi^n(y)\right) < \varepsilon$ for all $n \in \mathbb{Z}$.
- Every tds (X, φ) has a *maximal equicontinuous factor (MEF)* $(X_{\mathrm{eq}}, \varphi_{\mathrm{eq}})$, that is, an equicontinuous factor $(X_{\mathrm{eq}}, \varphi_{\mathrm{eq}})$ such that every other equicontinuous factor (Y, ψ) factors through $(X_{\mathrm{eq}}, \varphi_{\mathrm{eq}})$ with factor map $\pi = \tilde{\pi} \circ \pi_{\mathrm{eq}}$.

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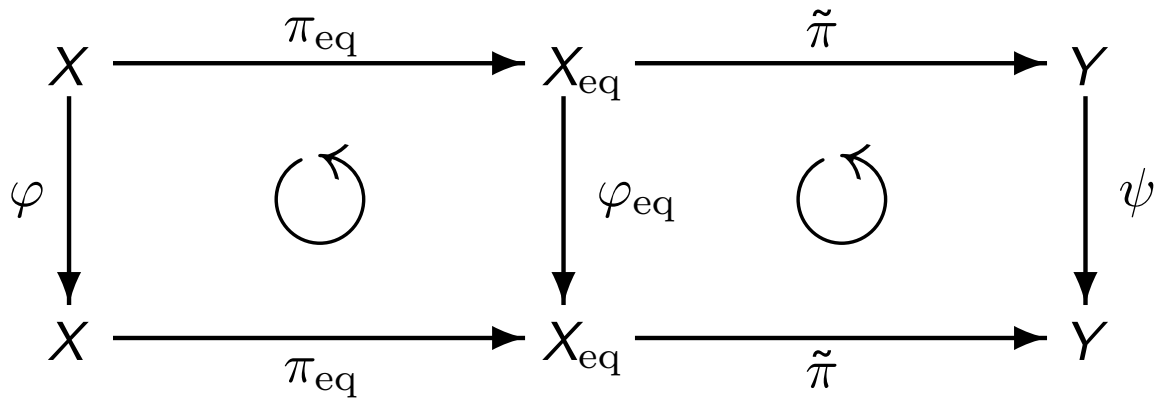
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Mean equicontinuous minimal systems

A tds (X, φ) is *mean equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$d(x, y) < \delta \quad \text{implies} \quad d_B^{\varphi}(x, y) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sum_{k=0}^{n-1} d\left(\varphi^k(x), \varphi^k(y)\right) < \varepsilon$$

d_B^{φ}: *Besicovitch (pseudo-)metric*

(X, φ) is called a *topo-isomorphic extension (of its MEF)* if it is uniquely ergodic and

$$\pi_{\mathrm{eq}} : (X, \mu, \varphi) \to (X_{\mathrm{eq}}, \mu_{\mathrm{eq}}, \varphi_{\mathrm{eq}})$$

is a measure-theoretic isomorphism.

Theorem (Downarowicz and Glasner, [LTY15, DG16, FGL22])

A minimal tds (X, φ) is mean equicontinuous if and only if it is a topo-isomorphic extension of its MEF.

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A minimal tds (X, φ) is mean equicontinuous if and only if it is a topo-isomorphic extension of its MEF.

Remarks on this structure result

- Topo-isomorphism to the MEF is equivalent to pure point dynamical spectrum with continuous eigenfunctions [FGL22] and implies zero topological entropy.
- Mean equicontinuous minimal systems include important example classes – e.g. Sturmian subshifts or regular Toeplitz flows and regular model sets – in particular in the context of aperiodic order.
- We call $x \in X$ an *injectivity point* if $\#\pi_{\mathrm{eq}}^{-1}(\pi_{\mathrm{eq}}(x)) = 1$. Similarly, we say $y \in X_{\mathrm{eq}}$ has a *singleton fibre* if $\#\pi_{\mathrm{eq}}^{-1}(y) = 1$.
- Topo-isomorphy to the MEF does not imply existence of injectivity points [DG16, HJ25]! (Although they exist in most examples.)
- By topo-isomorphy, there exist full measure sets $X^0 \subseteq X$, $X_{\mathrm{eq}}^0 \subseteq X_{\mathrm{eq}}$ such that $\pi_{\mathrm{eq}} : X^0 \to X_{\mathrm{eq}}^0$ is bijective. The inverse $g_0 : X_{\mathrm{eq}}^0 \to X^0$ can be extended to a measurable map $g : X_{\mathrm{eq}} \to X$ such that

$$g \circ \varphi_{\mathrm{eq}} \;=\; \varphi \circ g \quad \text{and} \quad g_* \mu_{\mathrm{eq}} \;=\; \mu$$

The map g can be chosen such that $g(X_{\mathrm{eq}})$ is φ-invariant.

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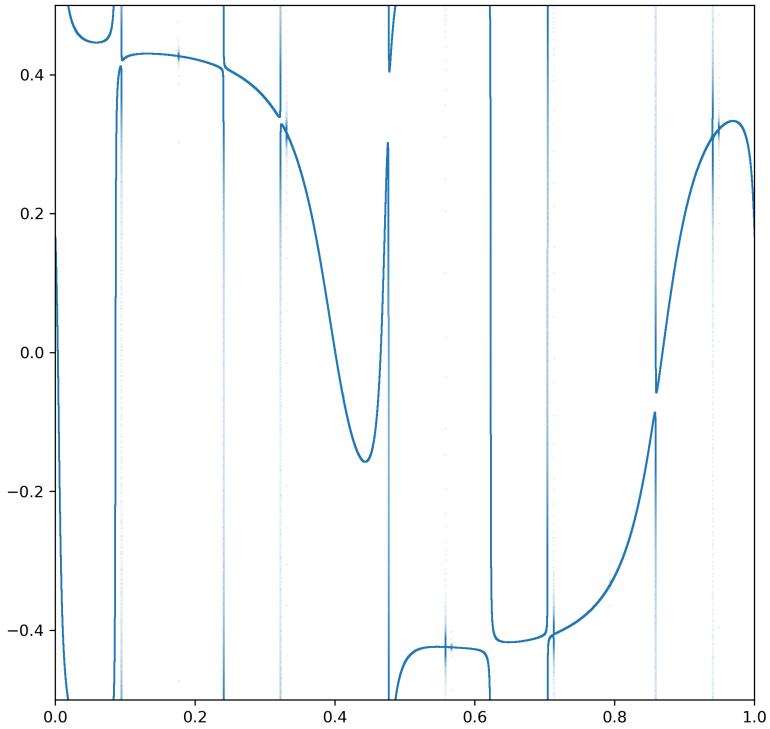
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A visualisation of the topo-isomorphic structure

Particular examples of m.e. minimal system are skew products on $\mathbb{T}^2$ with a unique dense invariant graph g that carries the unique inv. measure.

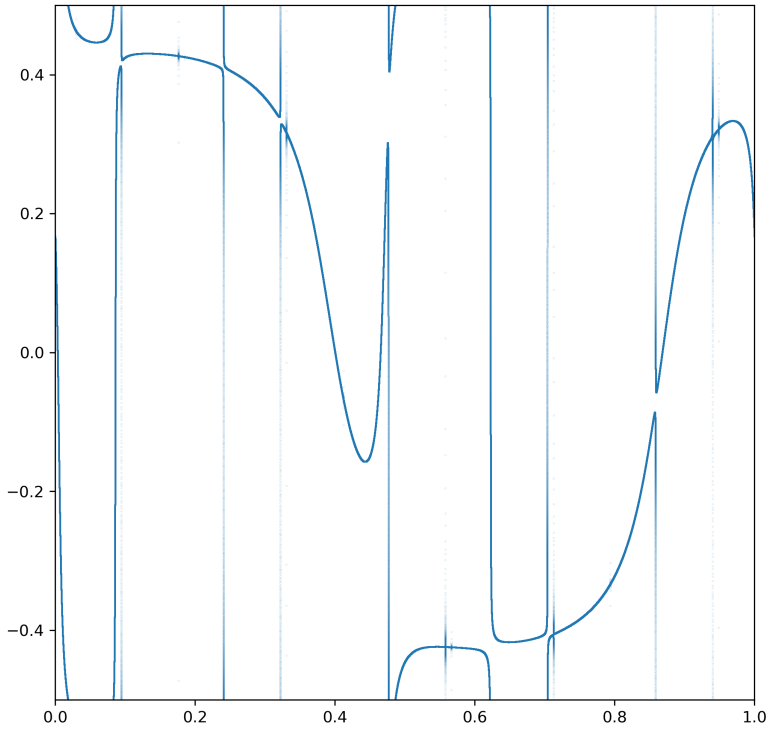


$$\varphi : \mathbb{T}^2 \to \mathbb{T}^2 \quad , \quad (x_1, x_2) \mapsto (x_1 + \alpha, \varphi_{x_1}(x_2))$$

Provides intuition for *'topo-isomorphy $\Rightarrow$ mean equicontinuity'*.

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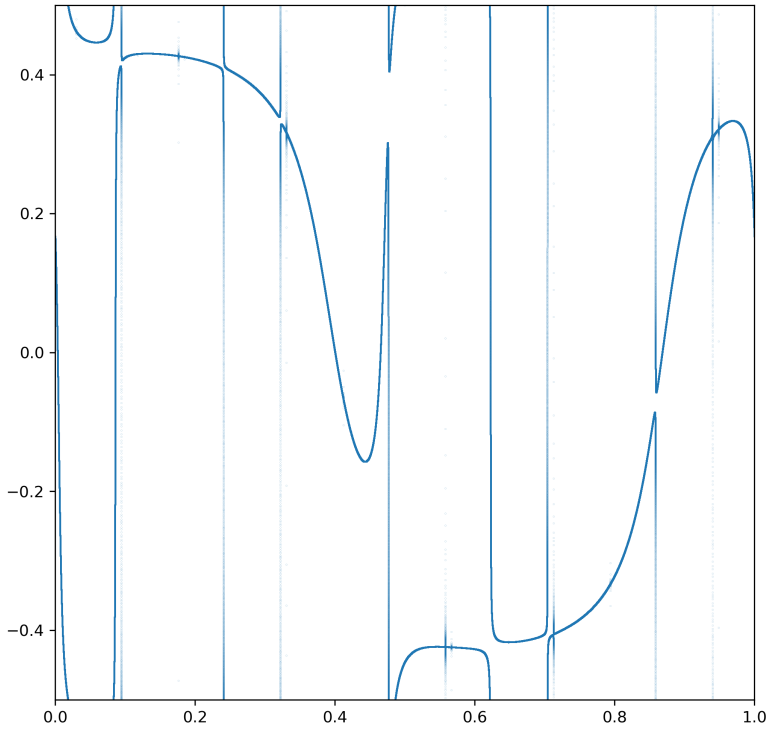


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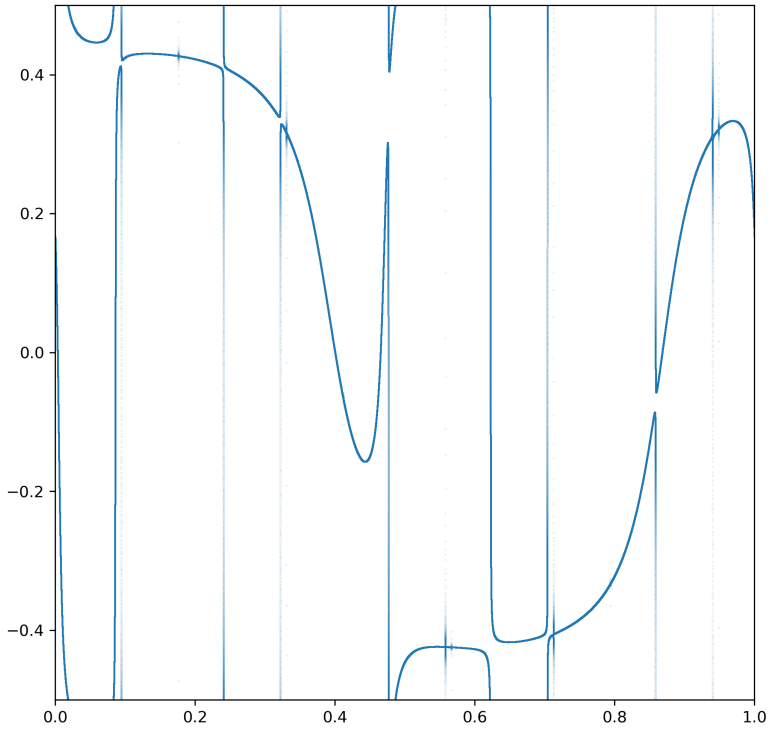


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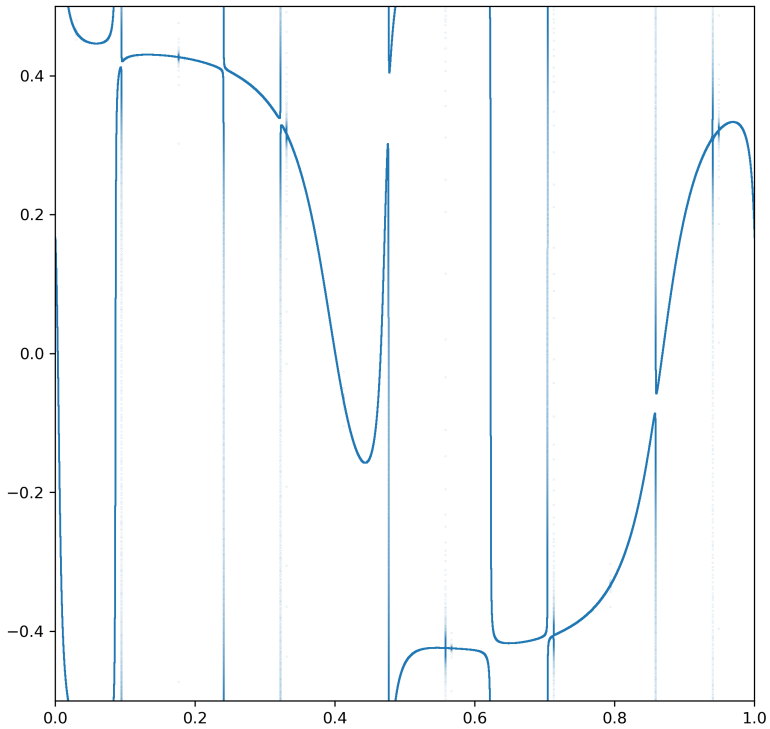
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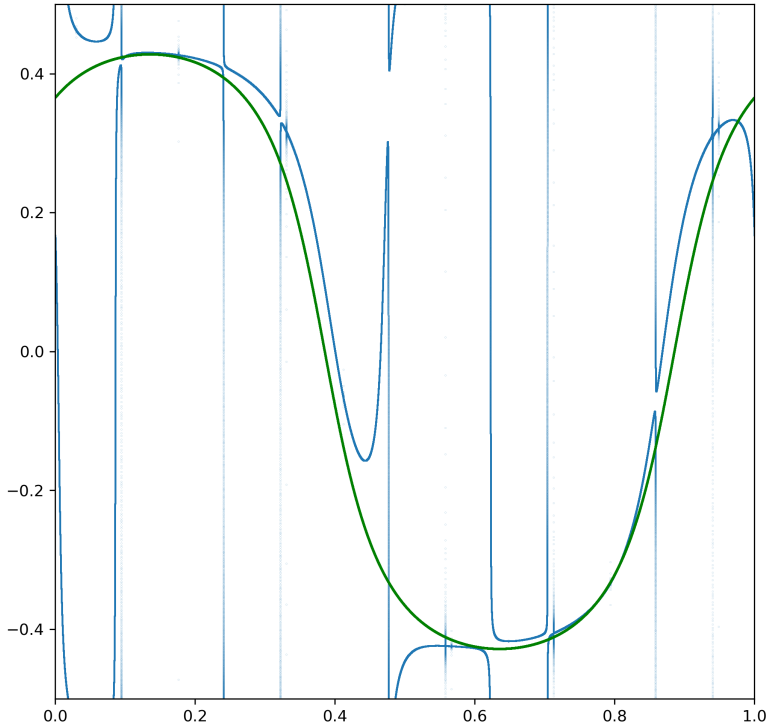
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Why topo-isomorphy implies mean equicontinuity



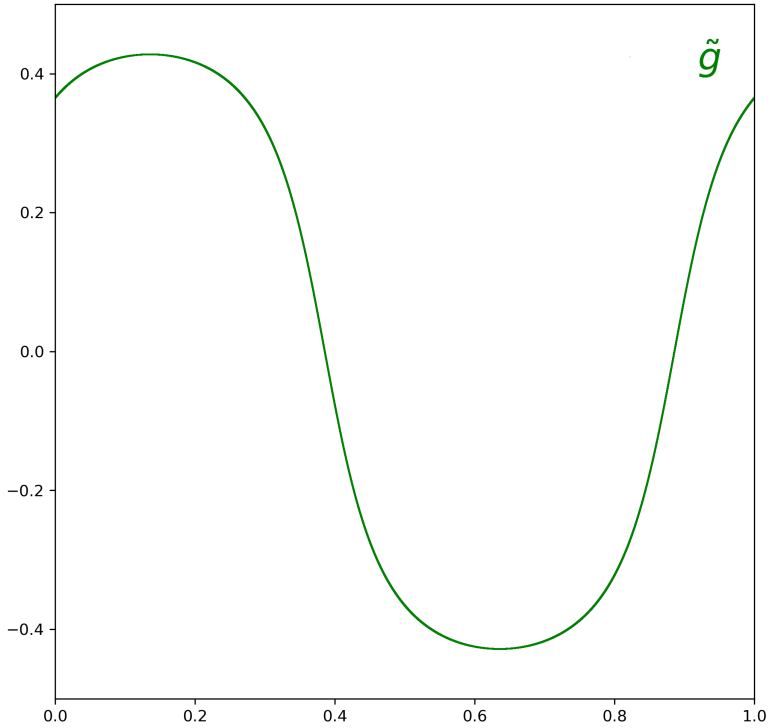
Consider the unique invariant graph g.

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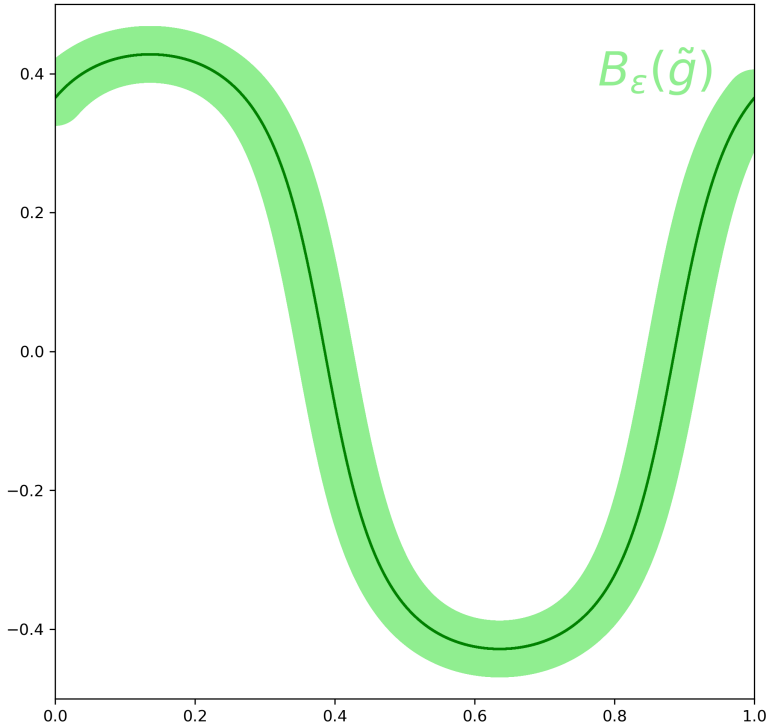
Approximate g with a cts graph $\tilde{g}$ that is ε-close on a set of measure $1 - \varepsilon$. This is possible e.g. due to Lusin's Theorem.

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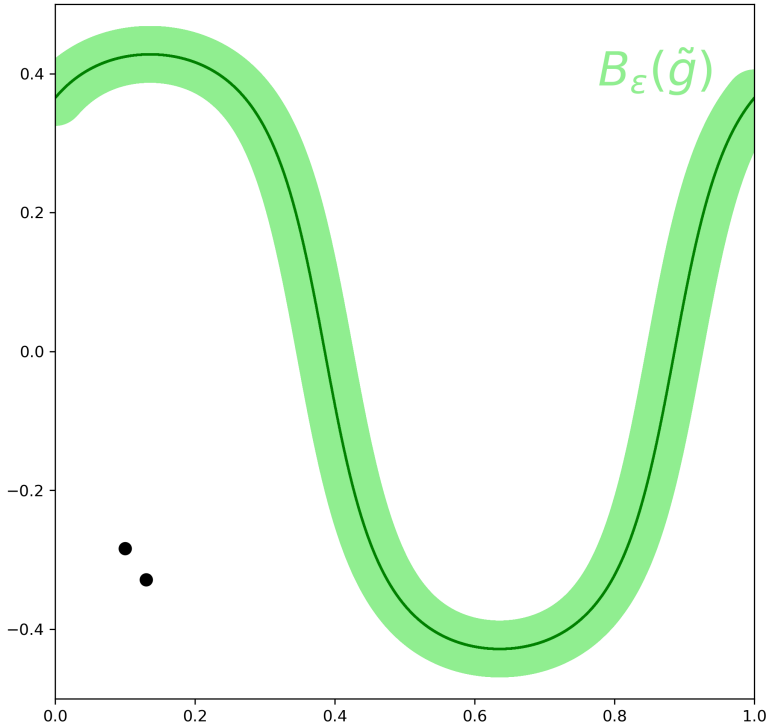
Now focus only on $\tilde{g}$.

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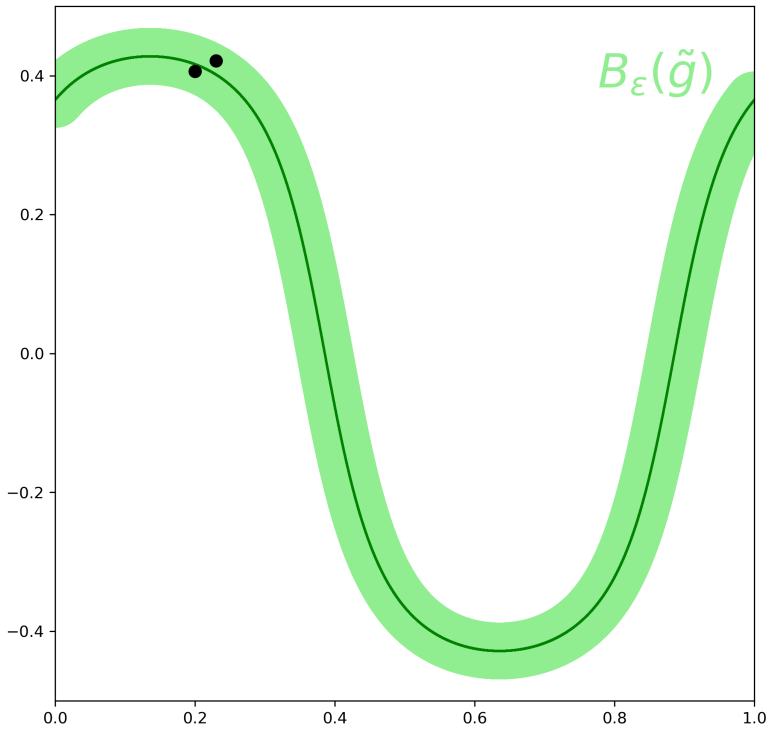
The ε-neighbourhood $B_\varepsilon(\tilde{g})$ has measure $> 1 - \varepsilon$. Due to unique ergodicity, any orbit must visit it with frequency $> 1 - \varepsilon$.

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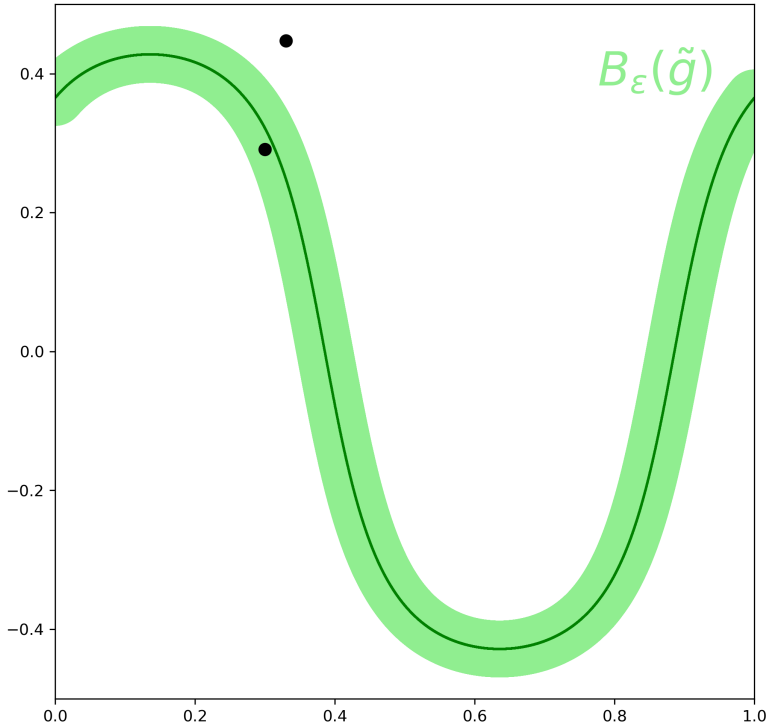
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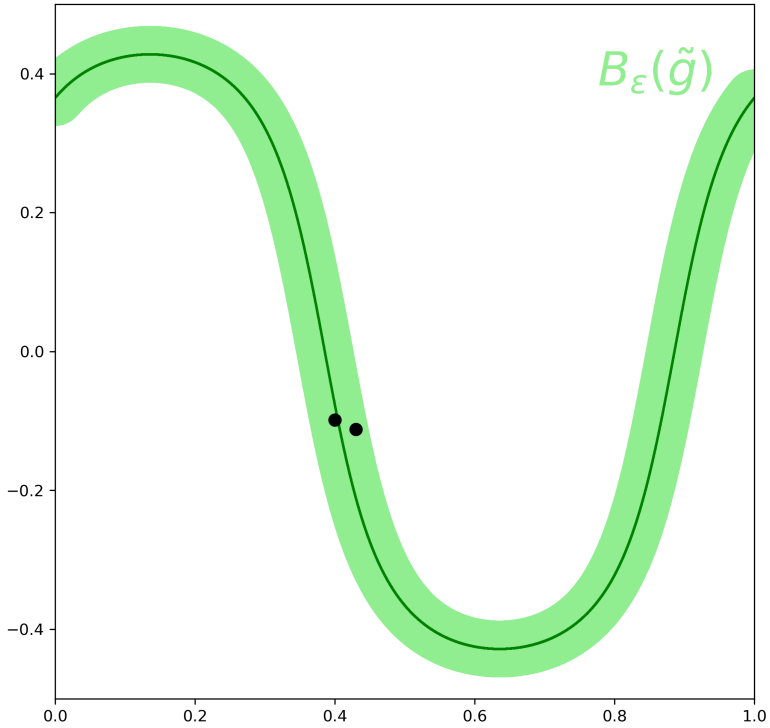
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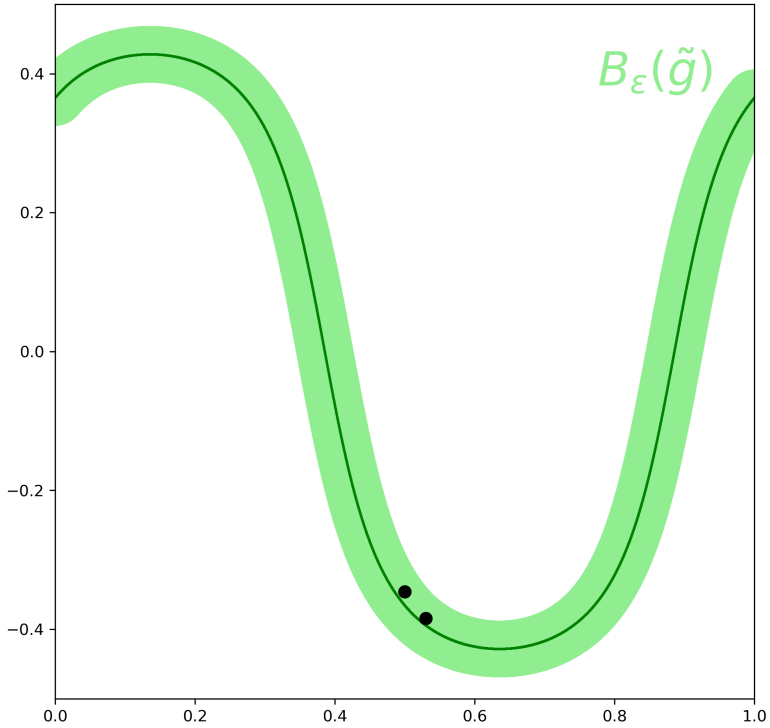
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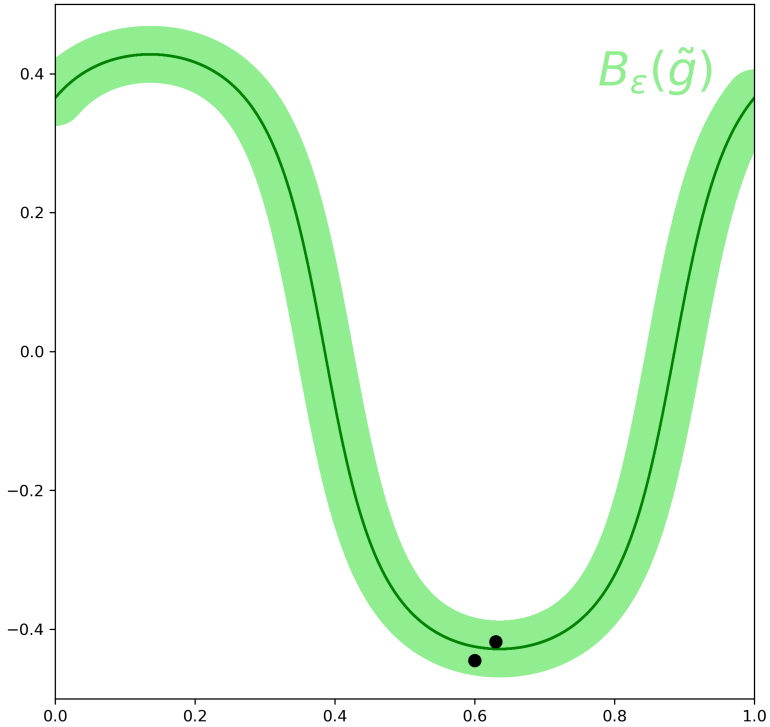
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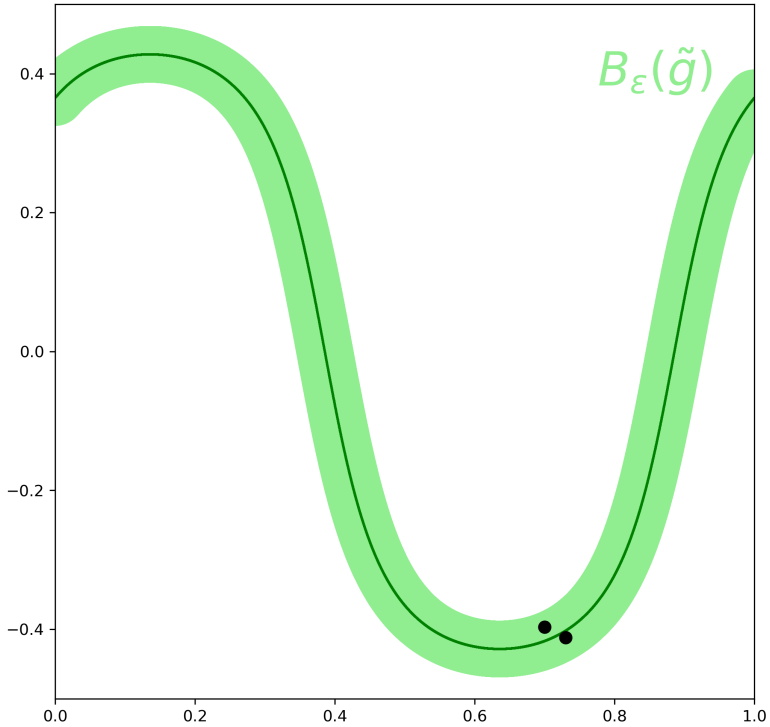
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Subclasses of mean equicontinuous systems

A tds (X, φ) is an *almost 1-1 extension (of its MEF)* if the set

$$\mathcal{I}(X) \;=\; \left\{ y \in X_{\mathrm{eq}} \mid \#\pi_{\mathrm{eq}}^{-1}(y) = 1 \right\} \;\subseteq\; X_{\mathrm{eq}}$$

is non-empty/residual. We also say it is *almost automorphic* in this case.

(X, φ) is called *regular* if $\mu_{\mathrm{eq}}(\mathcal{I}(X)) = 1$ (with μ_{eq} Haar measure on X_{eq}).

These notions define important subclasses of mean equicontinuous minimal systems. Both can be characterised in dynamical terms.

(X, φ) is called *diam mean equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $B \subseteq X$

$$\operatorname{diam}(B) < \delta \quad \text{implies} \quad \overline{\lim_{n\to\infty}} \sum_{k=0}^{n-1} \operatorname{diam}\left(\varphi^k(B)\right) < \varepsilon$$

Theorem (García-Ramos/Ye/J., [GRJY21])

A minimal system (X, φ) is diam mean equicontinuous iff it is regular.

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(X, φ) is called *frequently stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that for all $B \subseteq X$

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where $\overline{D}$ denotes the upper asymptotic density (upper Banach density).

Theorem (García-Ramos/Ye/J., [GRJY21])

A m.e. minimal system (X, φ) is almost automorphic iff it is frequently stable.

This result has been extended beyond the mean equicontinuous setting by Hu and Xu by considering the induced hyperspace dynamics

$$\hat{\varphi} : \mathcal{K}(X) \to \mathcal{K}(X), \ K \mapsto \varphi(K)$$

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A set $S \subseteq \mathbb{Z}$ is called an *independence set* for a pair of subsets $U_0, U_1 \subseteq X$ if

$$\forall a \in \{0,1\}^S \ \exists x_a \in X : \quad \Phi^s(x_a) \in U_{a_s} \quad \text{for all } s \in S \ .$$

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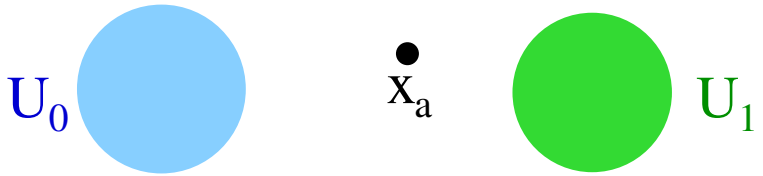
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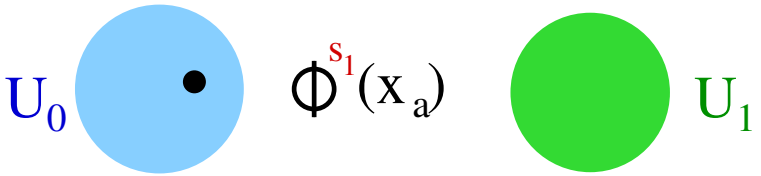
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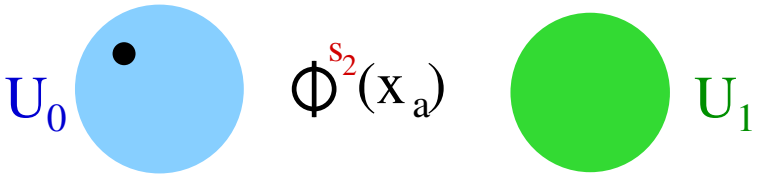
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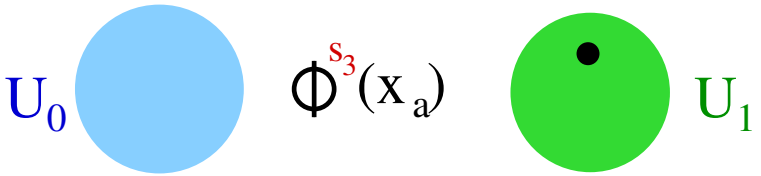
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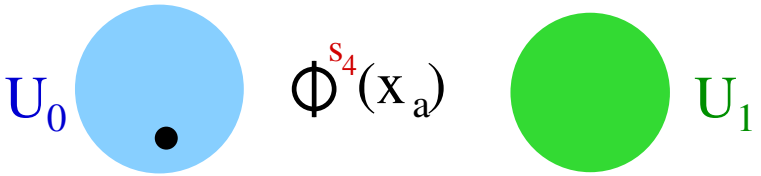
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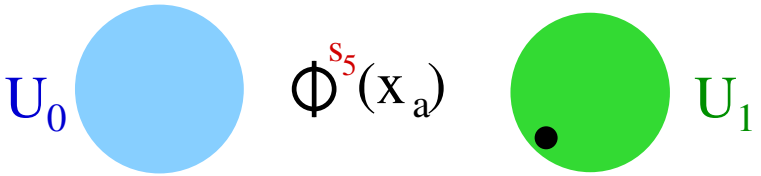


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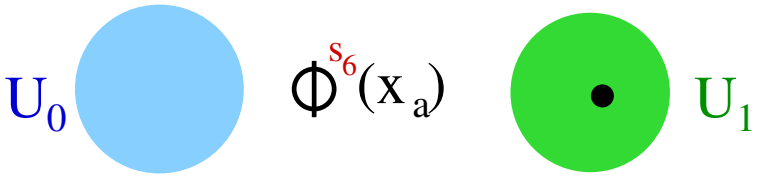
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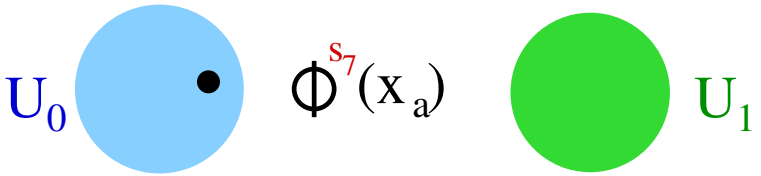
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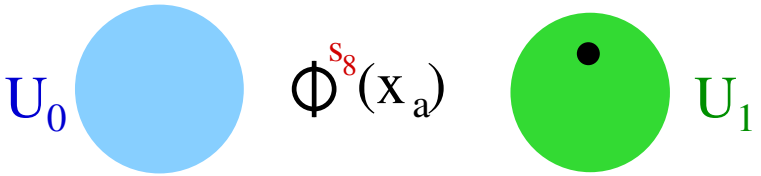


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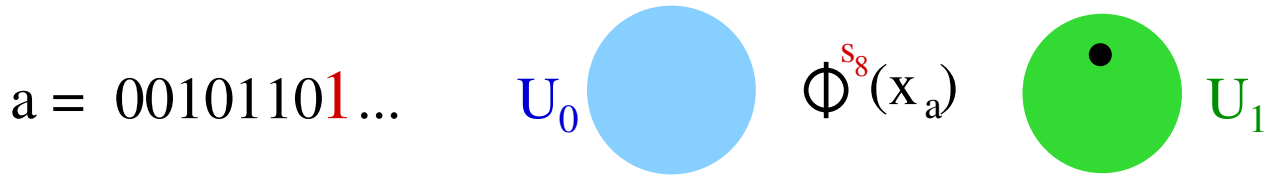
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A pair (x_0, x_1) is called an *IT pair* if $x_0 \neq x_1$ and for every pair of neighbourhoods U_i of x_i, there exists an infinite independence set S.

A tds (X, φ) is called *tame* if it does not admit any IT pair [KL07].

Theorem (Fuhrmann/Glasner/Oertel/J., [FGJO21])

For minimal tds, tame implies regular.

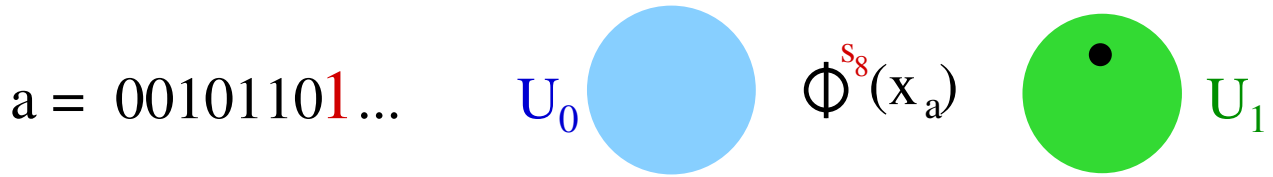
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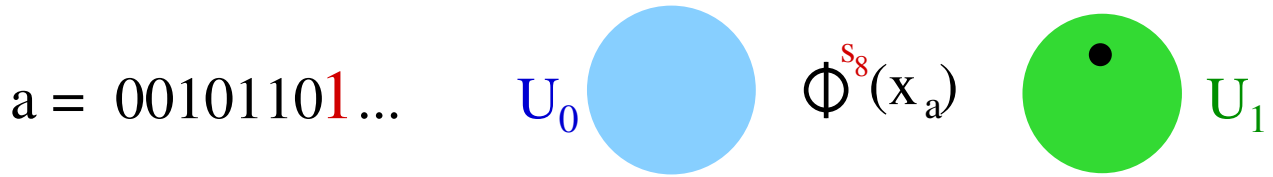
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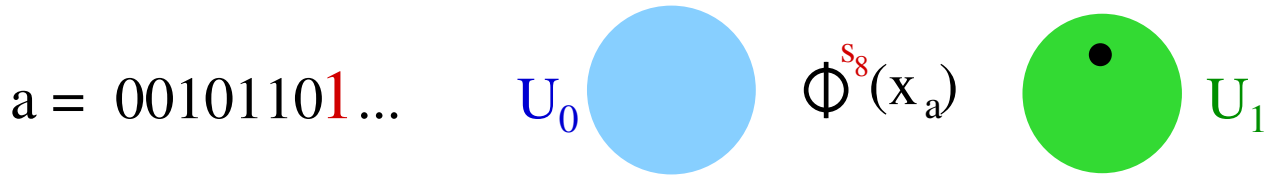
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Theorem (Fuhrmann/Glasner/Oertel/J., [FGJO21])

For minimal tds, tame implies regular.

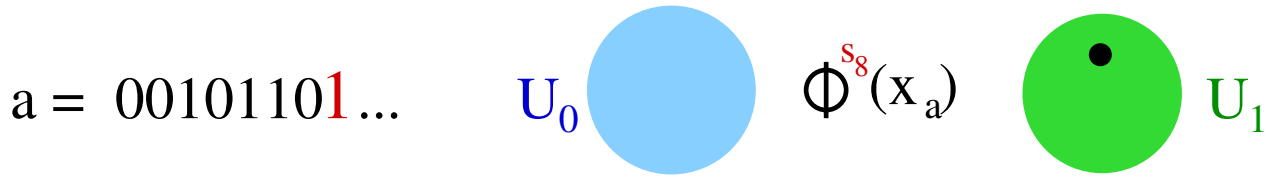
Builds on previous work by Glasner showing that minimal & tame implies almost automorphic [Gla18]. Works for amenable groups.

Tame systems

A set $S \subseteq \mathbb{Z}$ is called an *independence set* for a pair of subsets $U_0, U_1 \subseteq X$ if

$$\forall a \in \{0,1\}^S \; \exists x_a \in X : \quad \Phi^s(x_a) \in U_{a_s} \quad \text{for all } s \in S \; .$$

Let $S = \{s_1, s_2, \ldots\}$



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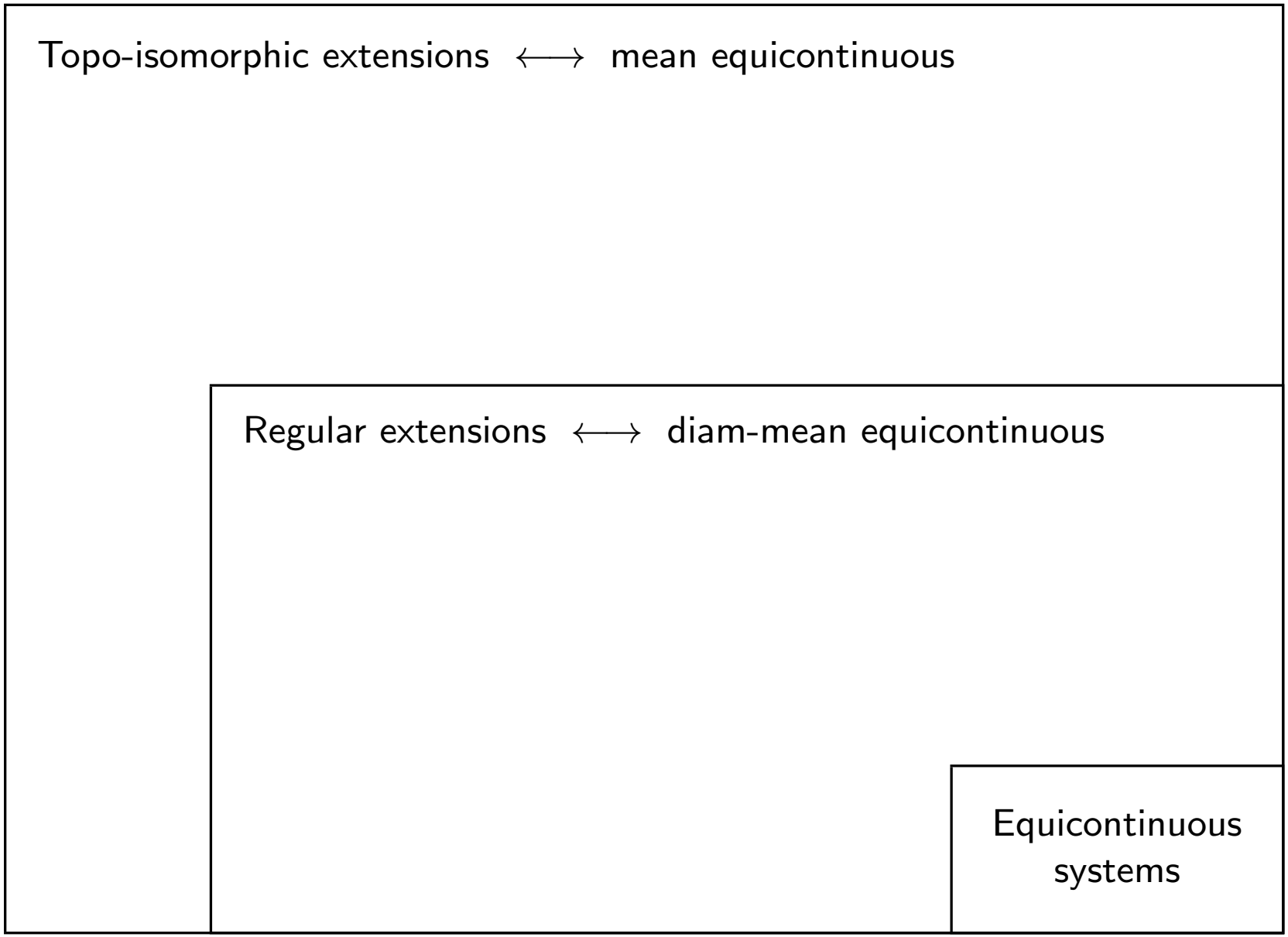
Hierarchy of mean equicontinuous systems

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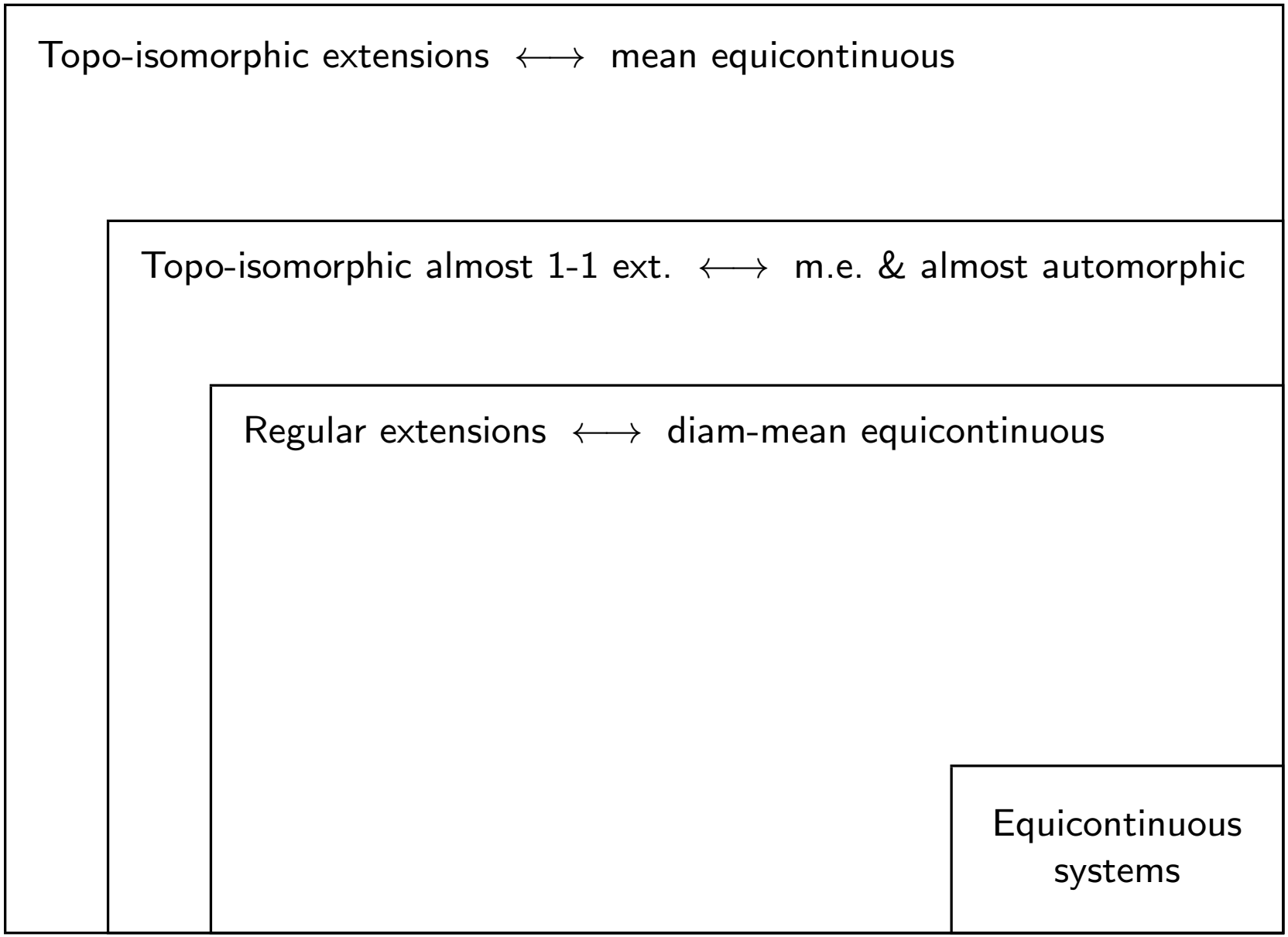
Topo-isomorphic extensions $\longleftrightarrow$ mean equicontinuous

Equicontinuous systems

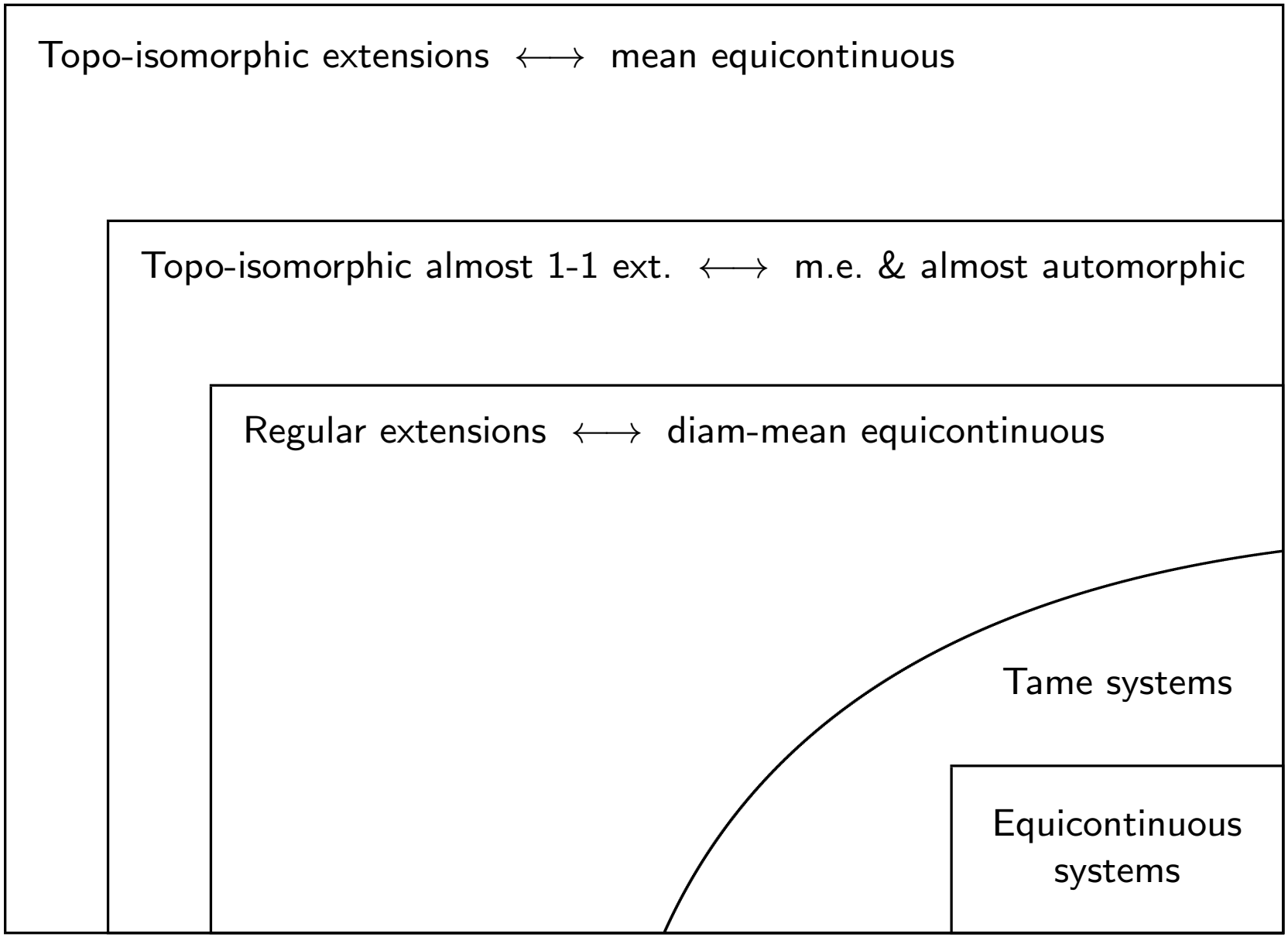
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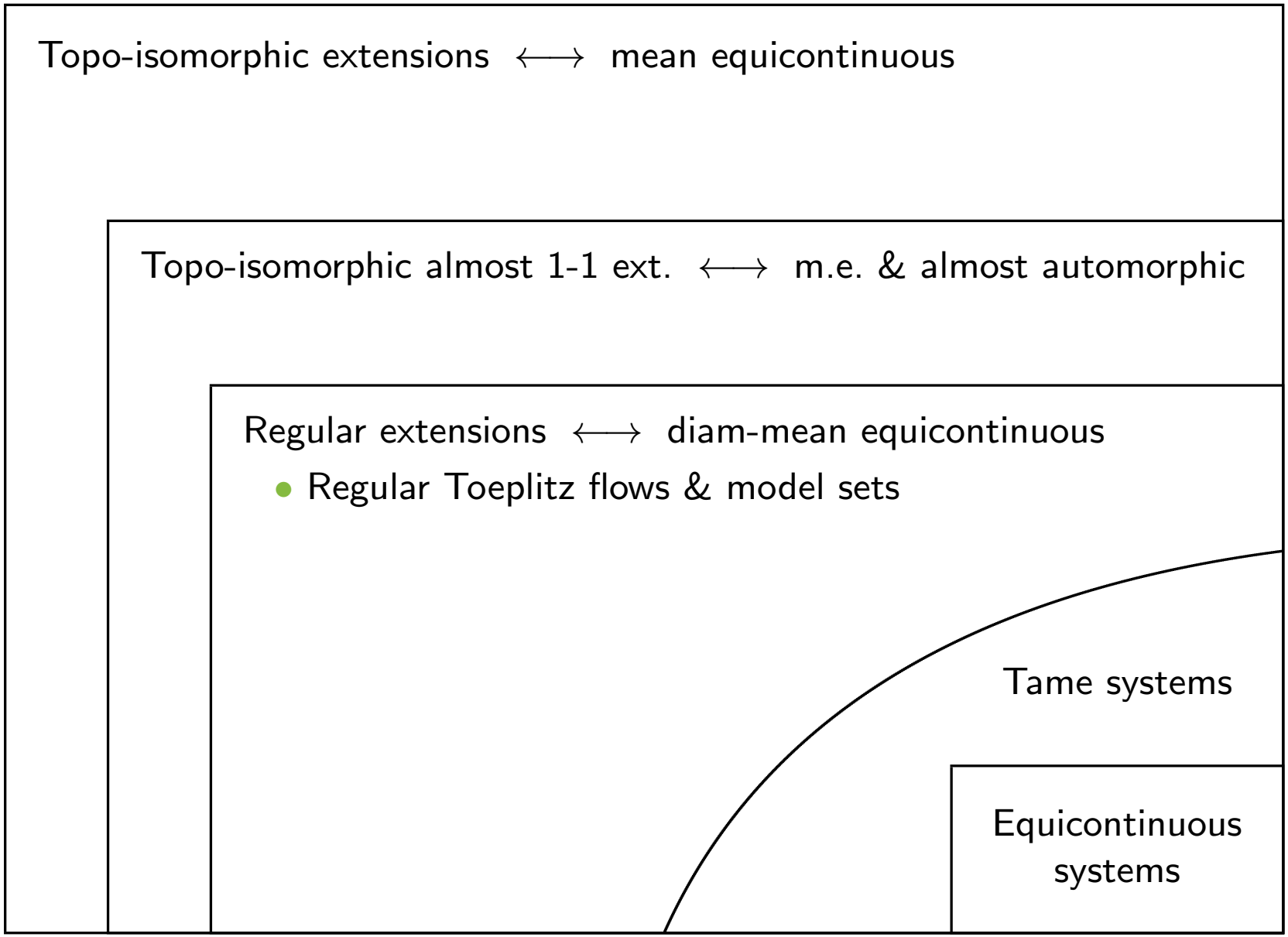
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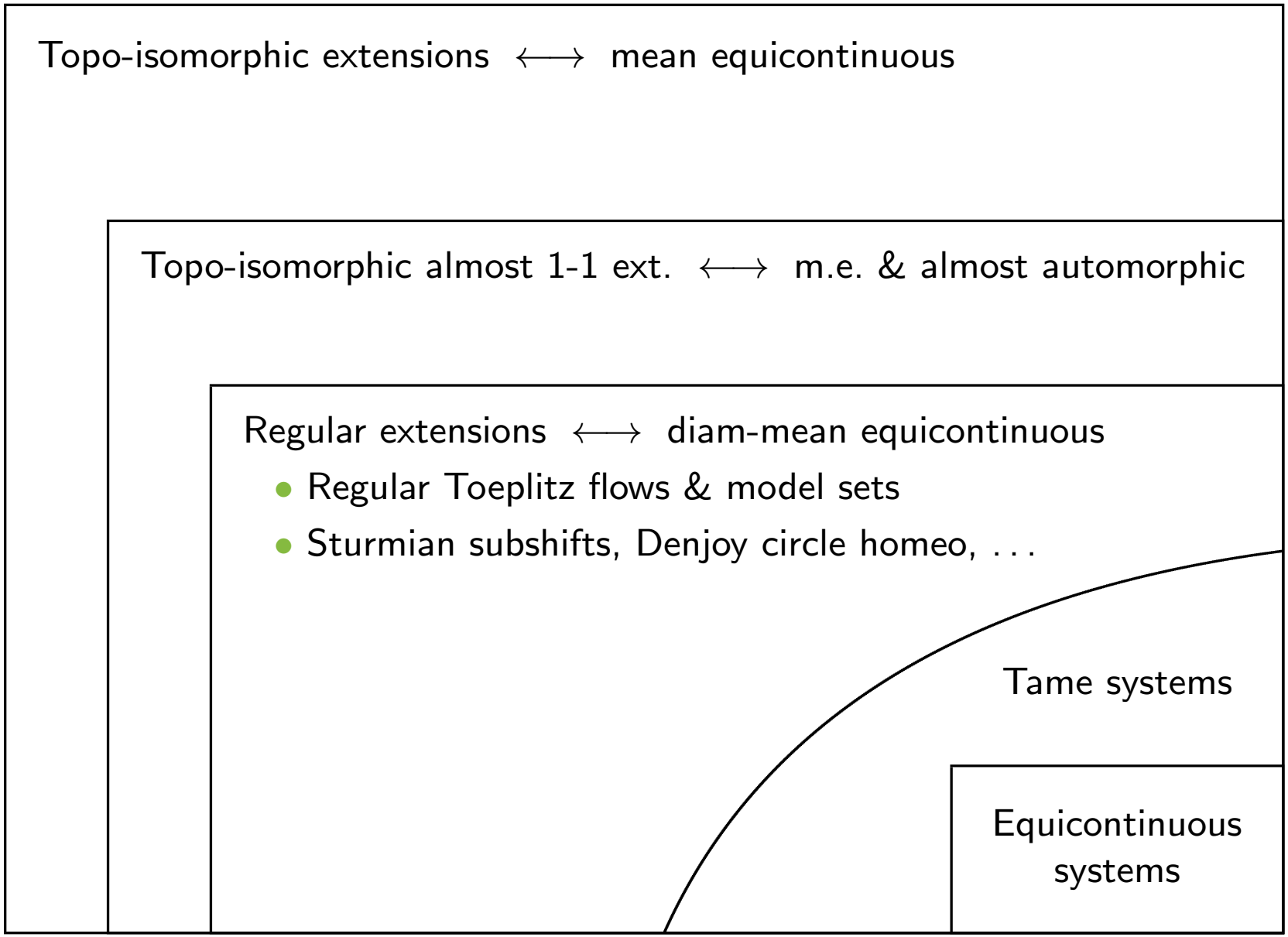
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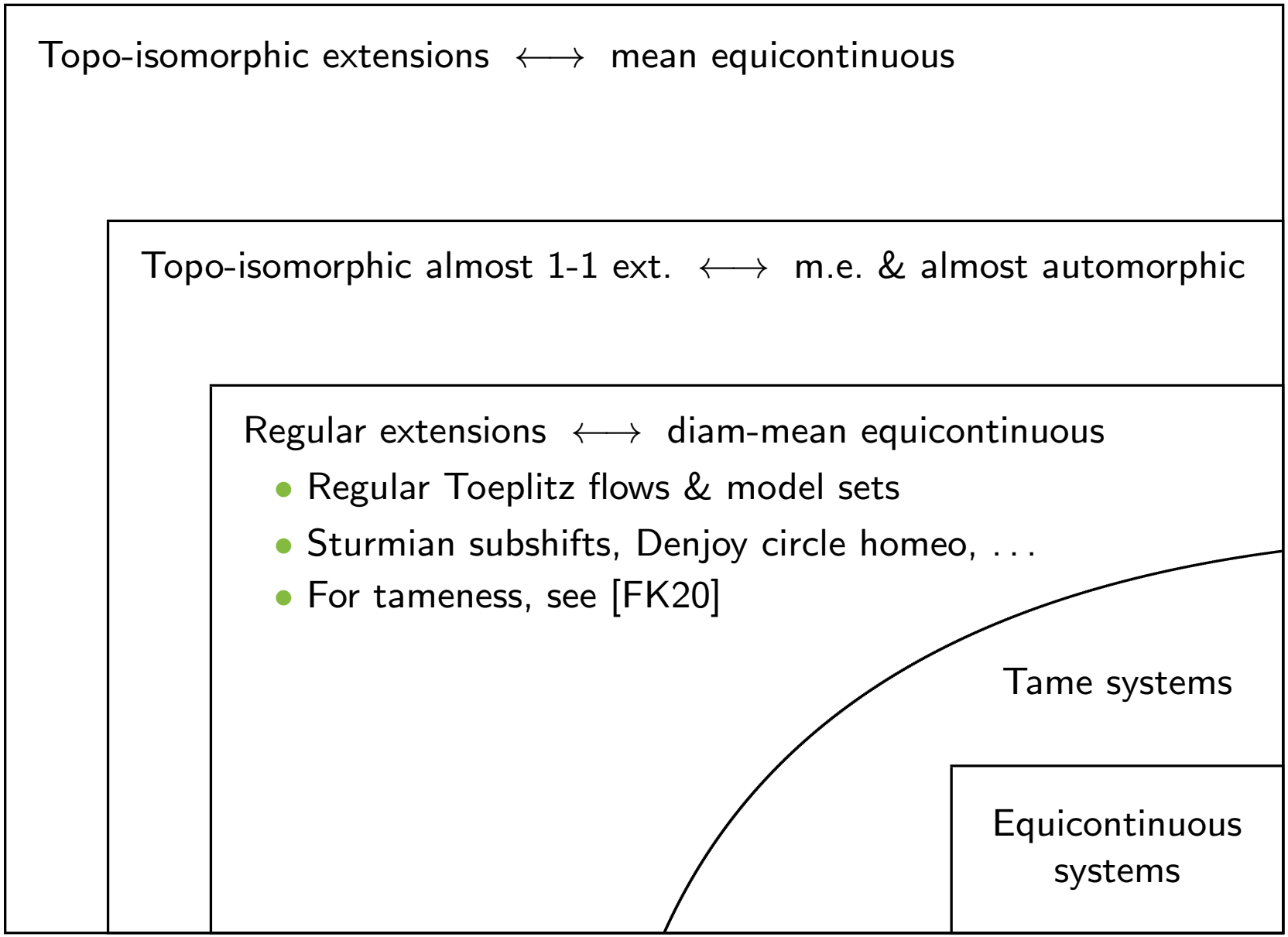
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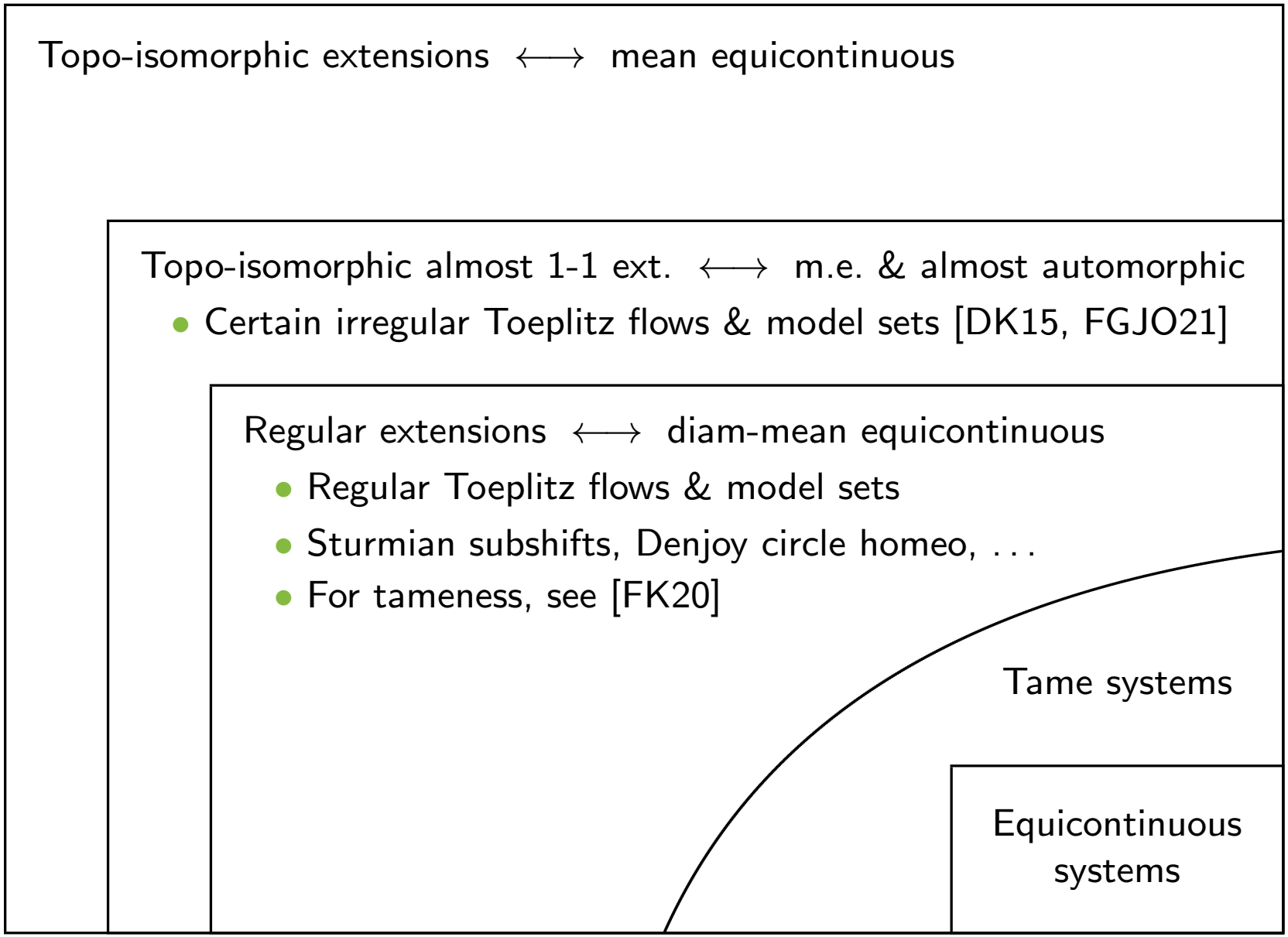
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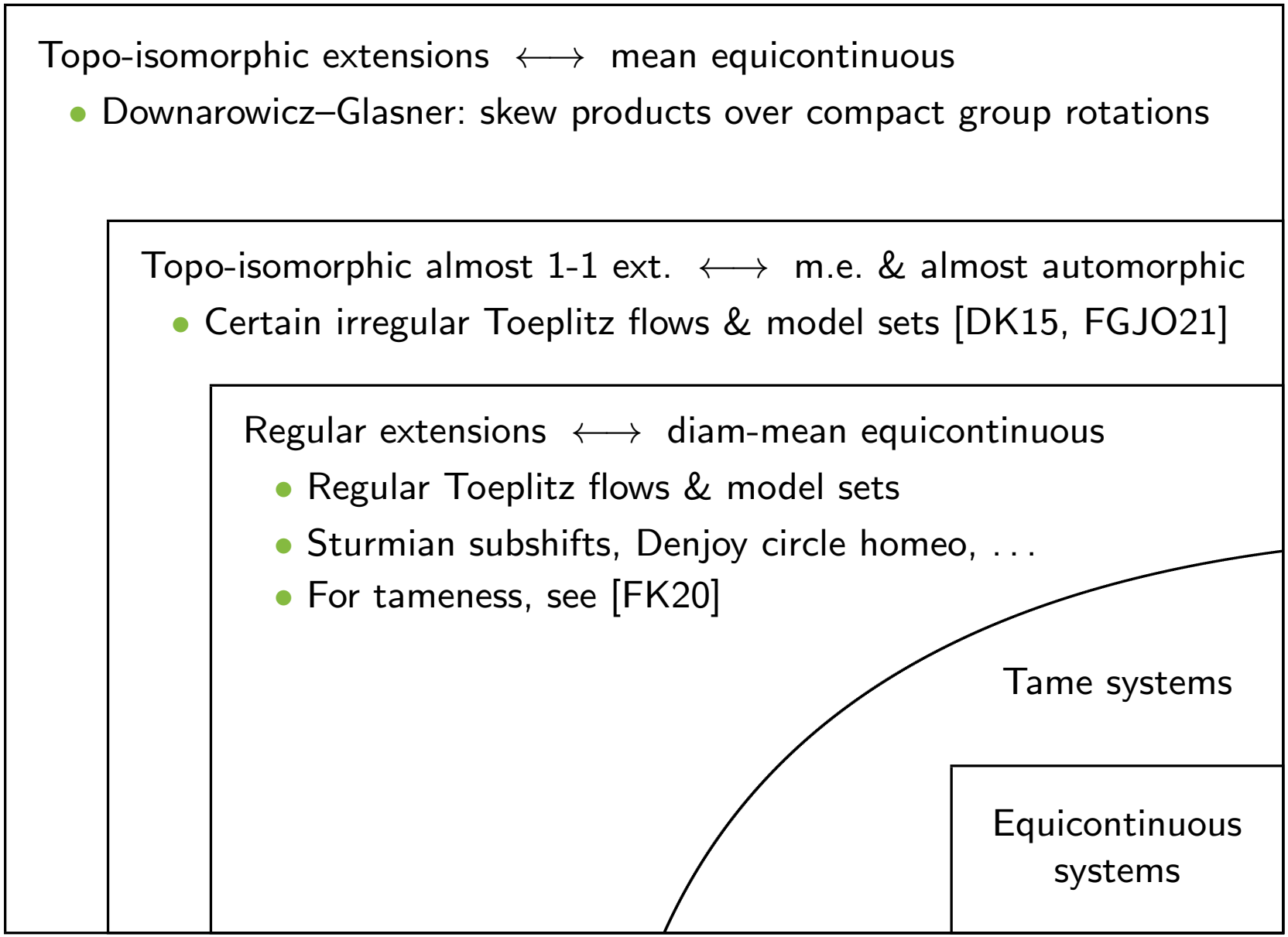
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Topo-isomorphic extensions $\longleftrightarrow$ mean equicontinuous

- Downarowicz–Glasner: skew products over compact group rotations
- Haupt/J.: smooth skew products on $\mathbb{T}^2$ via Anosov-Katok

Topo-isomorphic almost 1-1 ext. $\longleftrightarrow$ m.e. & almost automorphic

- Certain irregular Toeplitz flows & model sets [DK15, FGJO21]

Regular extensions $\longleftrightarrow$ diam-mean equicontinuous

- Regular Toeplitz flows & model sets
- Sturmian subshifts, Denjoy circle homeo, ...
- For tameness, see [FK20]

Tame systems

Equicontinuous systems

Amorphic complexity: a taylor-made top. invariant

Asymptotic separation frequencies: $x, y \in X,\ \delta > 0$

$$\nu_\delta^\varphi(x,y) = \overline{\lim_{n\to\infty}} \frac{1}{n} \sharp \left\{ 0 \le k < n \mid d\left(\varphi^k(x), \varphi^k(y)\right) \ge \delta \right\}$$

Separation numbers:

$$\mathrm{Sep}(\varphi, \delta, \nu) = \sup\{\sharp S \mid S \ \nu\text{-separated wrt } \nu_\delta^\varphi\}$$

finite separation numbers: $\forall \delta, \nu > 0 :\ \mathrm{Sep}(\varphi, \delta, \nu) < \infty$ for all $\delta, \nu > 0$

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Amorphic complexity [FGJ16]:

$$\mathrm{ac}(\varphi) = \sup_{\delta>0} \lim_{\nu \searrow 0} \frac{\log \mathrm{Sep}(\varphi, \delta, \nu)}{-\log \nu}$$

If the limit does not exist, define upper and lower ac $(\overline{\mathrm{ac}}(\varphi), \underline{\mathrm{ac}}(\varphi))$.

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Theorem (Fuhrmann/Gröger/Kwietniak/J., [FGJK23])

A minimal tds (X, φ) has finite separation numbers if and only if it is mean equicontinuous.

In many cases, precise values or at least estimates of amorphic complexity are available. This includes

- regular Toeplitz flows [FGJ16]
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In all cases, the respective formulas involve dimensional quantities.

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Amorphic complexity of symbolic systems

If (X, d) is an ultrametric space, then the ν_δ^φ are pseudo-metrics which induce the same topology as the Besicovitch pseudo-metric d_B^φ.

We let $x \sim y :\Leftrightarrow d_B^\varphi(x, y) = 0$ and denote by $[X] = X/\sim$ the corresponding quotient space (*Besicovitch space*).

Proposition ([FGJ16, FG20])

In the above situation, we have

$$\overline{\mathrm{ac}}(X, \varphi) \;=\; \overline{\mathrm{Dim}}_B([X]) \;,$$

where $\overline{\mathrm{Dim}}_B$ *denotes the upper box dimension.*

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Amorphic complexity and constant length substitutions

- Let $\mathcal{A}$ be a finite alphabet, $\mathcal{A}^*$ the set of finite words in $\mathcal{A}$ and denote by $|w|$ the length of a finite word in $\mathcal{A}$.
- Suppose that $\vartheta : \mathcal{A} \to \mathcal{A}^*$ a primitive substitution rule and denote by X_ϑ the induced (strictly ergodic) subshift.
- ϑ is of *constant length* is there exists $k \in \mathbb{N}$ with $|\vartheta(a)| = k$ for all $a \in \mathcal{A}$. In this case, let $|\vartheta| = k$.
- If $\mathcal{A} = \{0, 1\}$, denote by $|\vartheta|_*$ the number of mismatches between $\vartheta(0)$ and $\vartheta(1)$.
- Example: For the period doubling substitution

$$0 \mapsto 01 \quad , \quad 1 \mapsto 00 \ ,$$

we have $|\vartheta|_* = 1$.

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Theorem (Fuhrmann and Gröger, [FG20])

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$$\operatorname{ac}(X_\vartheta, \sigma) \;=\; \frac{\log |\vartheta|}{\log |\vartheta| - \log |\vartheta|_*} \,.$$

The result relies on the fact that the Besicovitch space $[X_\vartheta]$ equals the attractor of the IFS $(F_j = \sigma^j \circ \vartheta)_{j=0,\dots,|\vartheta|-1}$. In general, we have

Theorem (Fuhrmann and Gröger, [FG20])

Suppose that ϑ is a primitive constant length substitution with discrete dynamical spectrum. Then the following hold.

(i) $\operatorname{ac}(X_\vartheta, \sigma) = 0$ *iff* (X_ϑ, σ) *is periodic;*

(ii) $1 \leq \operatorname{ac}(X_\vartheta, \sigma) < \infty$ *iff* (X_ϑ, σ) *has pure point spectrum;*

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Let $\mathcal{A} = \{0, 1\}$ and suppose that ϑ is a primitive constant length substitution and X_ξ is infinite. Then

$$\operatorname{ac}(X_\vartheta, \sigma) \;=\; \frac{\log |\vartheta|}{\log |\vartheta| - \log |\vartheta|_*} \,.$$

The result relies on the fact that the Besicovitch space $[X_\vartheta]$ equals the attractor of the IFS $(F_j = \sigma^j \circ \vartheta)_{j=0,\dots,|\vartheta|-1}$. In general, we have

Theorem (Fuhrmann and Gröger, [FG20])

Suppose that ϑ is a primitive constant length substitution with discrete dynamical spectrum. Then the following hold.

(i) $\operatorname{ac}(X_\vartheta, \sigma) = 0$ *iff* (X_ϑ, σ) *is periodic;*

(ii) $1 \leq \operatorname{ac}(X_\vartheta, \sigma) < \infty$ *iff* (X_ϑ, σ) *has pure point spectrum;*

(iii) $\operatorname{ac}(X_\vartheta, \sigma) = \infty$ *iff* (X_ϑ, σ) *has a continuous spectral component.*

Constant length substitutions and inflation tilings

Similar results are obtained by Baake, Gähler and Gohlke in [BGG25] for inflation tilings.

In [GK26], these results are extended to the case of arbitrary finite alphabets and a precise formula for primitive constant length substitutions with pp spectrum is established. It is of the form

$$\mathrm{ac}(X_\vartheta, \sigma) \;=\; \frac{\log|\vartheta|}{\log|\vartheta| - \log\lambda^s}\;,$$

where λ^s is the leading eigenvalue of the substitution matrix of an associated *'discrepancy substitution'*. Moreover

Theorem (Gröger and Krawczyk, [GK26])

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The multivariate case

Why multivariate?

- Mean equicontinuous systems cover a broad range of important classes, in particular in aperiodic order and low complexity dynamics.
- Nevertheless, paradigmatic examples like Thue-Morse, and other interesting classical and recent examples, are not included.
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Characterisation of finite-to-one topomorphic extensions

Given $m \in \mathbb{N}$, we say (X, φ) is an *$m:1$ topomorphic extension (of if its MEF)* if m is the least integer such that there exists a measurable set $M \subseteq X$ such that

(i) $\mu(M) = 1$ for all φ-invariant Borel measures on X;

(ii) $\#\left(M \cap \pi_{\mathrm{eq}}^{-1}(y)\right) \leq m$ for all $y \in X_{\mathrm{eq}}$.

When (X, φ) is an $m:1$ topomorphic extension, the set M above can be chosen as the graph image of an m-valued measureable function

$$g : X_{\mathrm{eq}} \to X^m \quad , \quad y \mapsto (g_1(y), \ldots, g_m(y))$$

such that

- $\pi_{\mathrm{eq}}(g_i(y)) = y$ for $i = 1, \ldots, m,\ y \in X_{\mathrm{eq}}$;
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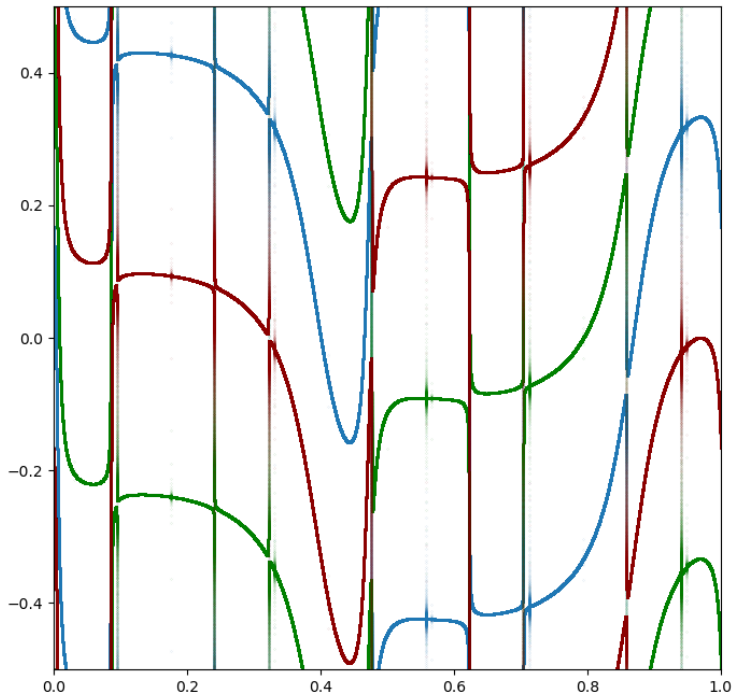
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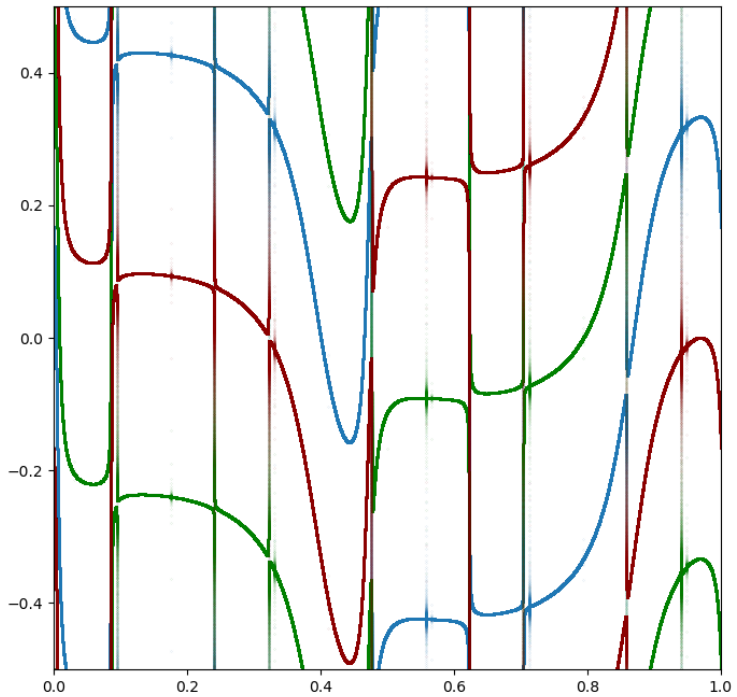
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In the skew product situation, an $m : 1$ topomorphic extension structure means that invariant measures are supported on a multivalued invariant graphs.



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For a dynamical characterisation, we consider the *'multidistance'*

$$D_m(x_1, \ldots, x_m) \;=\; \min_{1 \le i < j \le m} d(x_i, x_j)$$

and define the *Besicovitch m-distance* as

$$\overline{D}_m^{\varphi}(x_1, \ldots, x_m) \;=\; \overline{\lim_{n \to \infty}} \frac{1}{n} \sum_{k=0}^{n-1} D_m\left(\varphi^k(x_1), \ldots, \varphi^k(x_m)\right) \;.$$

Then (X, φ) is called *mean m-equicontinuous* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

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Theorem ([BHJ24])

If (X, φ) is a minimal $m:1$-topomorphic extension, then it is mean $(m+1)$-equicontinuous.

The argument still follows the same intuition as in [DG16] for the case $m = 1$. The converse implication is more intricate. Yet, we have

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Subclasses of multivariate mean equicontinuous systems

We say (X, φ) is an *almost $m:1$ extension (of its MEF)* if

$$m = \min\left\{\#\pi_{\mathrm{eq}}^{-1}(y) \mid y \in X_{\mathrm{eq}}\right\} .$$

Further, we let

$$\operatorname{diam}_m(B) = \sup_{x_1,\ldots,x_m \in B} D_m(x_1, \ldots, x_m)$$

and say (X, φ) is *frequently m-stable* if for all $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\operatorname{diam}_m(B) < \varepsilon \quad \text{implies} \quad \overline{D}\left(\{n \in \mathbb{N} \mid \operatorname{diam}_m(\varphi^n(B)) < \varepsilon\}\right) > 0 .$$

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Multivariate tameness

A set $S \subseteq \mathbb{Z}$ is called an *independence set* for a collection of subsets $U_0, \ldots, U_{m-1} \subseteq X$ if

$$\forall a \in \{0, \ldots, m-1\}^S \ \exists x_a \in X : \quad \Phi^s(x_a) \in U_{a_s} \quad \text{for all } s \in S \ .$$

A tuple $x_1, \ldots, x_{m-1}$ is called an *IT m-tuple* if $x_i \neq x_j$ whenever $i \neq j$ and for every collection of neighbourhoods U_i of x_i, there exists an infinite independence set S.

A tds (X, φ) is called *m-tame* if it does not admit any IT m-tuple.

Theorem (Liu, Xu and Zhang, [LXZ25])

If a minimal system (X, φ) is $(m+1)$-tame, then it is m-regular.

The proof is direct, without using almost m-automorphy, and gives a simpler proof of the *'tame implies regular'* result in the abelian case.

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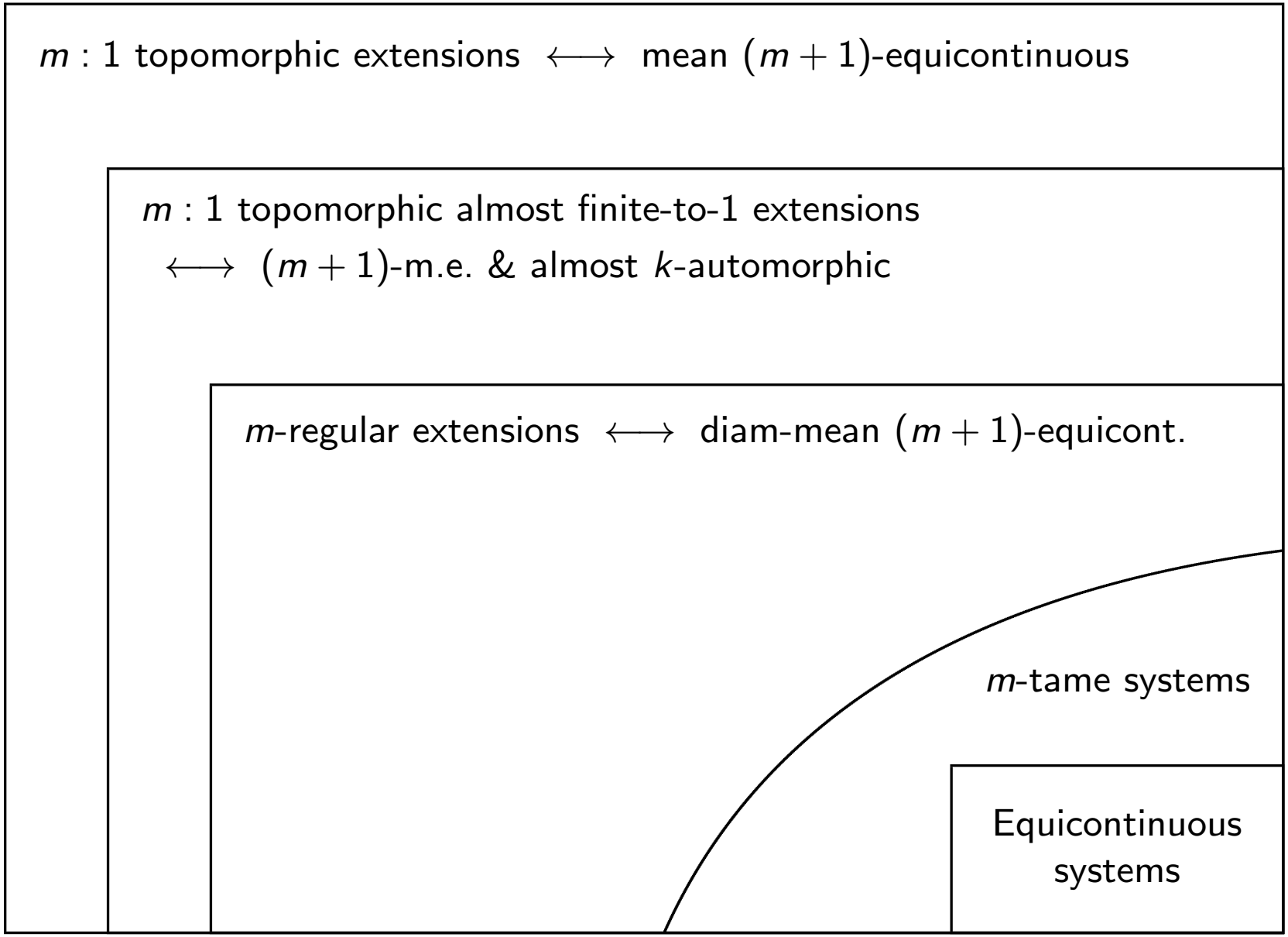
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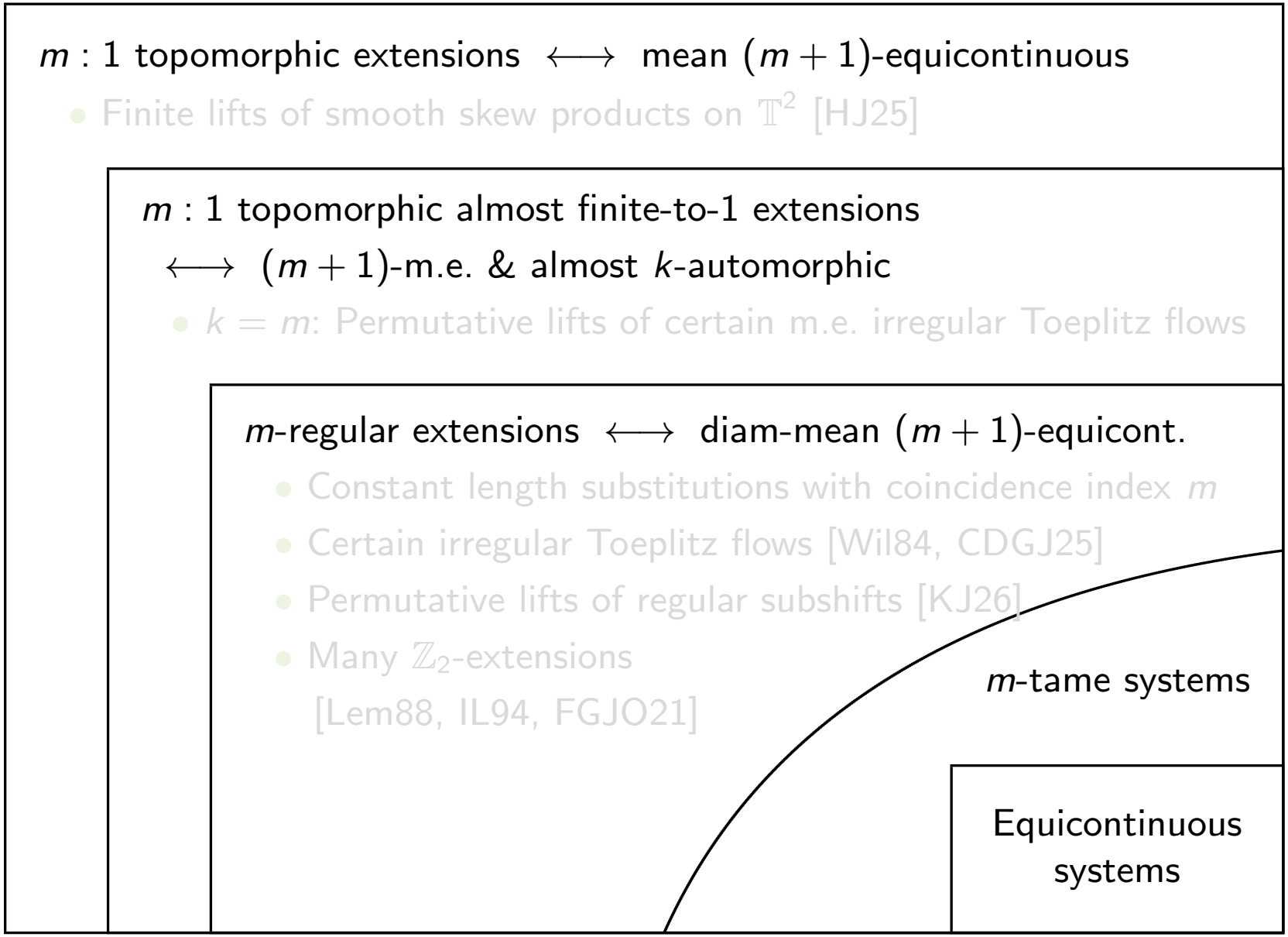
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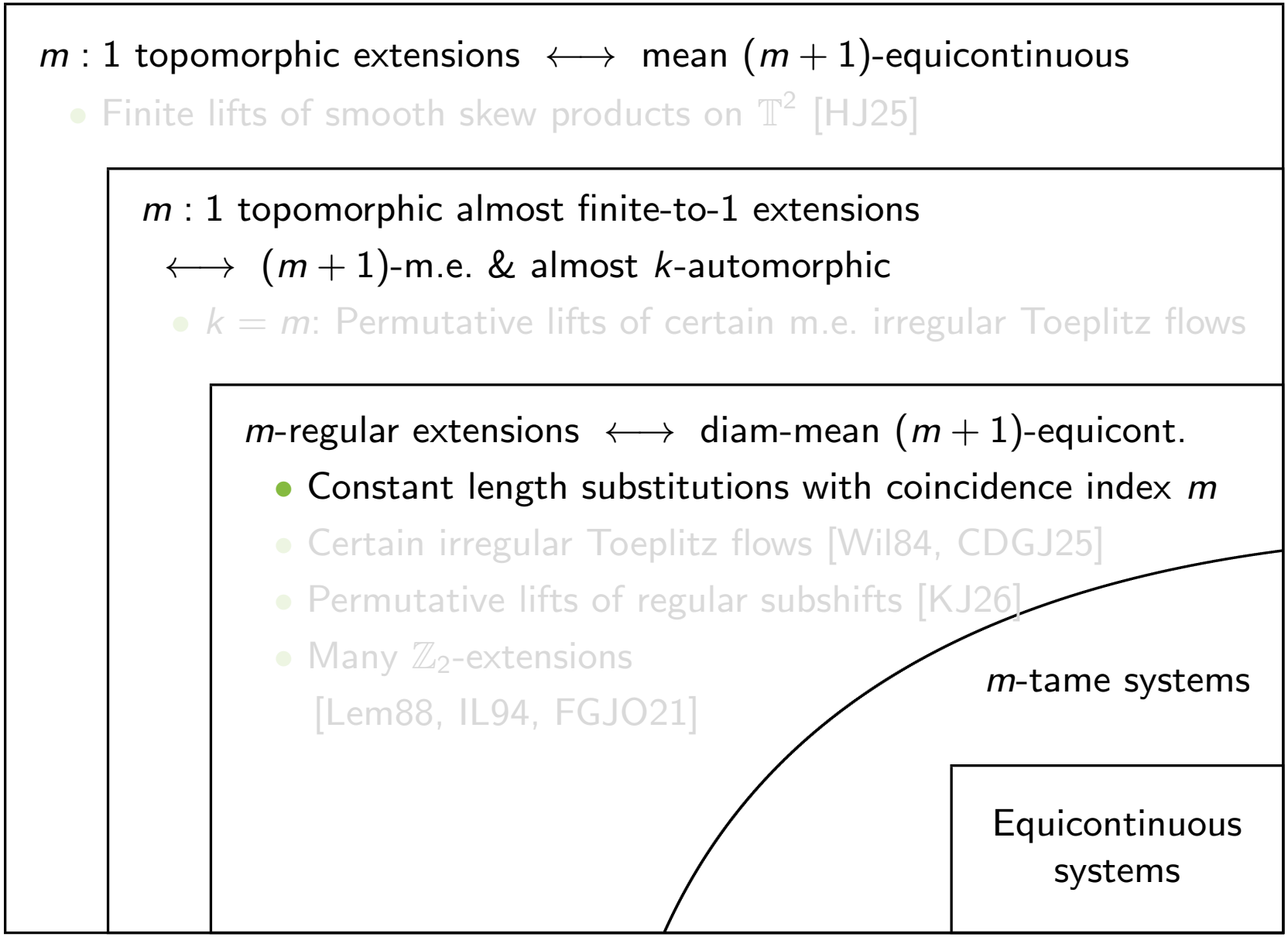
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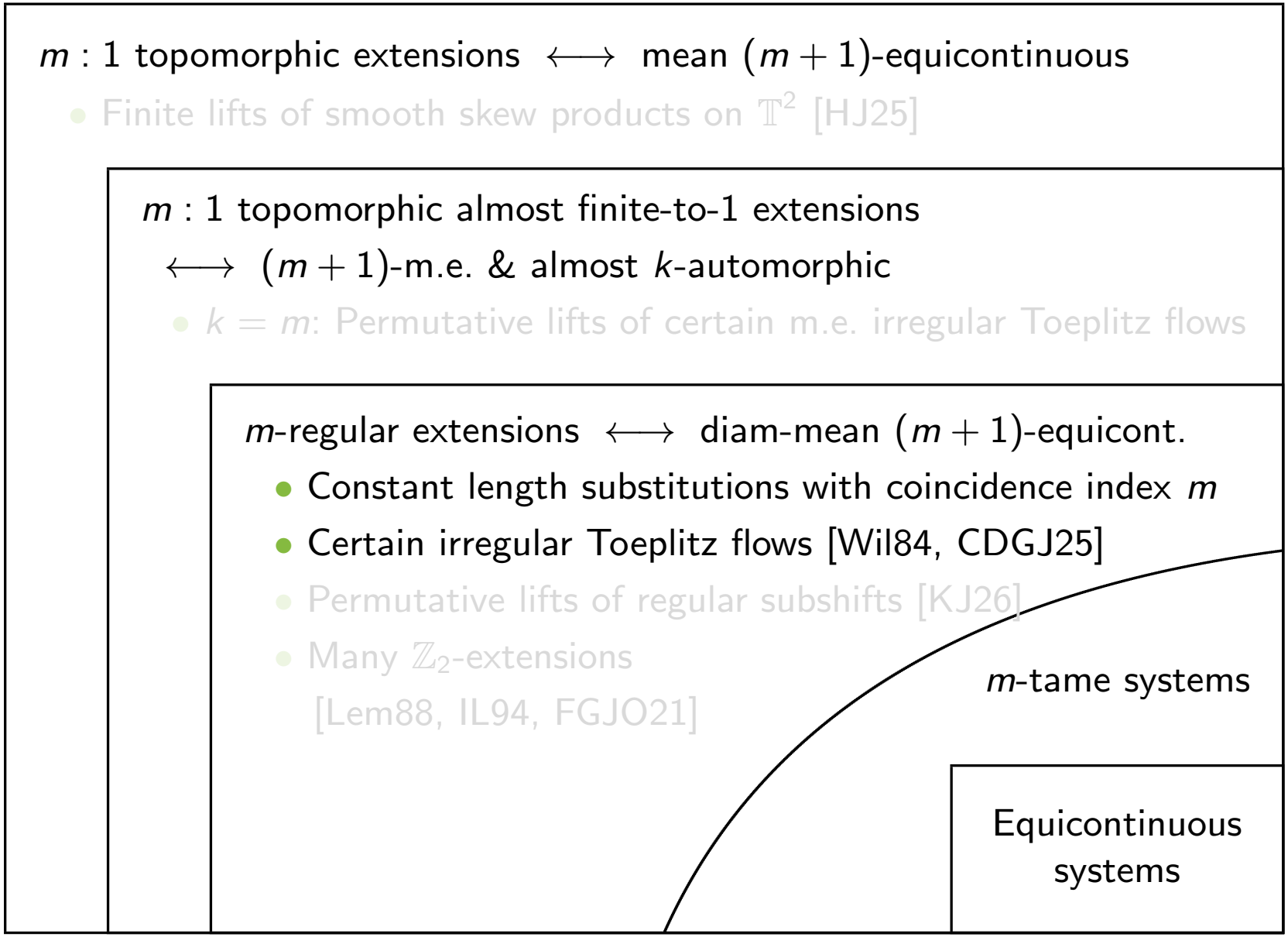
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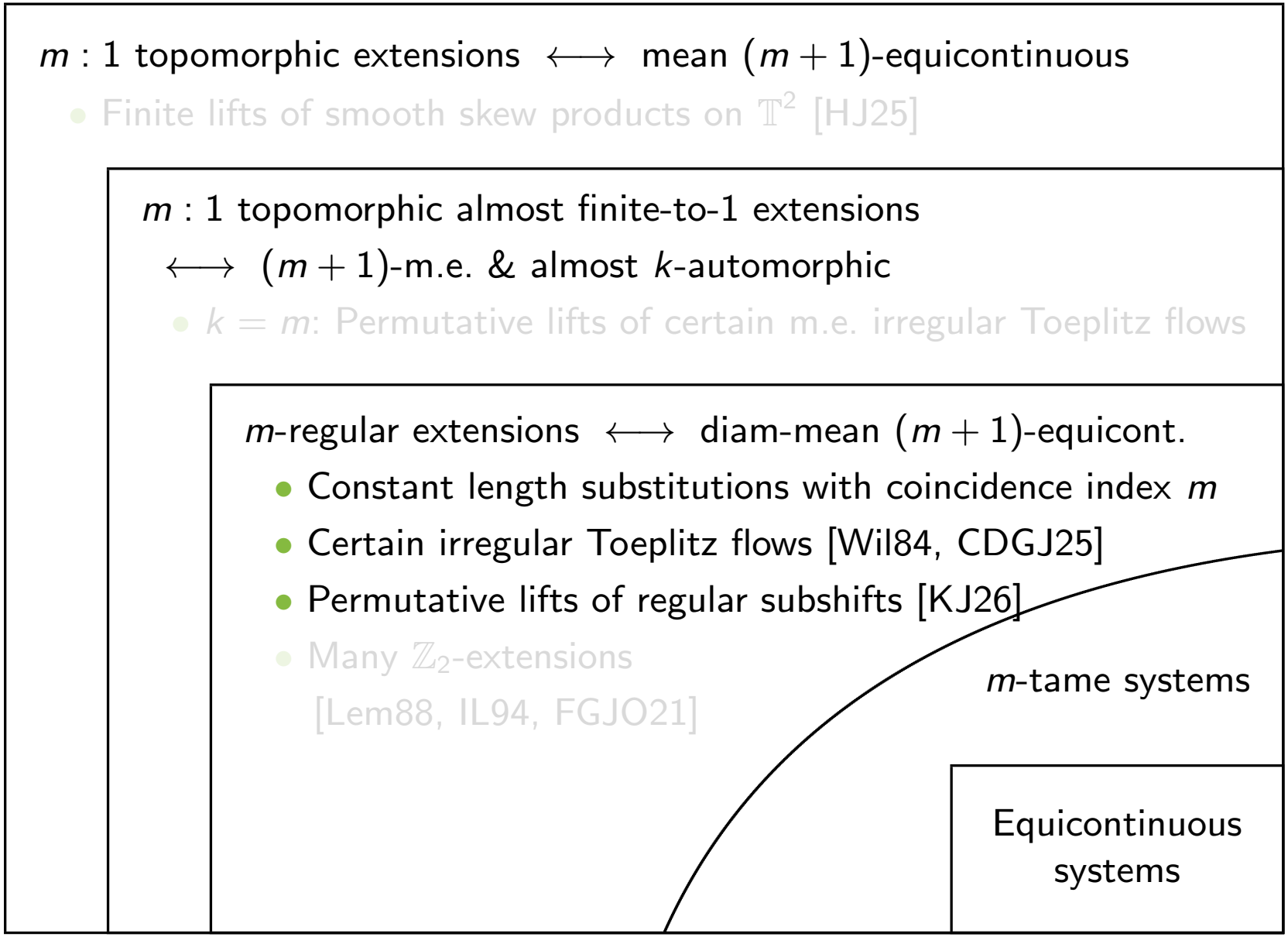
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- Finite lifts of smooth skew products on $\mathbb{T}^2$ [HJ25]

$m:1$ topomorphic almost finite-to-1 extensions

$\longleftrightarrow$ $(m+1)$-m.e. & almost k-automorphic

- $k=m$: Permutative lifts of certain m.e. irregular Toeplitz flows

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- Constant length substitutions with coincidence index m
- Certain irregular Toeplitz flows [Wil84, CDGJ25]
- Permutative lifts of regular subshifts [KJ26]
- Many $\mathbb{Z}_2$-extensions [Lem88, IL94, FGJO21]

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Perspectives

- Further dynamical properties and phenomena for finite-to-one topomorphic extensions
- $\rightarrow$ in particular spectral properties
- Taylor-made topological invariants for $m : 1$ topomorphic extensions:

  Topomorphic complexities

  ($\rightarrow$ work in progress with Lino Haupt and Maik Gröger)

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Koniec

Dziękuję za zaproszenie